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Geometric algebra techniques in flux compactifications (II)

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Abstract: We study constrained generalized Killing spinors over the metric cone and cylinder of a (pseudo-)Riemannian manifold, developing a toolkit which can be used to investigate certain problems arising in supersymmetric flux compactifications of supergravity theories. Using geometric algebra techniques, we give conceptually clear and computationally effective methods for translating supersymmetry conditions for the metric and fluxes of the unit section of such cylinders and cones into differential and algebraic constraints on collections of differential forms defined on the cylinder or cone. In particular, we give a synthetic description of Fierz identities, which are an important ingredient of such problems. As a non-trivial application, we consider the most general $\mathcal{N} = 2$ compactification of eleven-dimensional supergravity on eight-manifolds.

Keywords: Flux compactifications, Differential and Algebraic Geometry, Supergravity Models, Classical Theories of Gravity

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Contents

1 Introduction

A central problem in the study of flux compactifications of supergravity and string theories is that of finding geometric descriptions of supersymmetry conditions for various backgrounds in the presence of fluxes. This leads to beautiful and highly non-trivial connections with various subjects in differential geometry [1–6]. As pointed out in [7], the general problem of re-formulating supersymmetry conditions for flux backgrounds admits a powerful resolution based on geometric algebra techniques [8–14], an approach which is highly advantageous from a conceptual and computational standpoint.

The purpose of this paper is to combine the methods and ideas of [7] with an extension of the cone formalism of [15], providing a re-formulation of the latter within the theory of

Kähler-Atiyah algebras and bundles and developing a toolkit which can be used to solve a series of problems arising in the study of certain classes of flux compactifications. In particular, we show how the geometric re-formulation of generalized Killing spinor equations which was given in [7] can be lifted from a (pseudo-)Riemannian manifold to its metric cylinder and metric cone — a correspondence which is of particular interest in certain situations when one cannot encode the supersymmetry conditions through a reduction of the structure group of the compactification manifold itself. As a non-trivial example, section 3 applies such techniques and results to the study of the most general flux compactifications of M-theory on eight-manifolds preserving $\mathcal{N} = 2$ supersymmetry in 3 dimensions — a class of solutions which was not analyzed in full generality before (our generalization compared to the celebrated work of [16] being that we do *not* impose any chirality conditions on the internal part of the supersymmetry generators). In that example, we have a single algebraic condition $Q\xi = 0$, with $Q = \frac{1}{2}\gamma^m\partial_m\Delta - \frac{1}{288}F_{mpqr}\gamma^{mpqr} - \frac{1}{6}f_p\gamma^p\gamma^{(9)} - \kappa\gamma^{(9)}$ and $A_m = \frac{1}{4}f_p\gamma_m{}^p\gamma^{(9)} + \frac{1}{24}F_{mpqr}\gamma^{pqr} + \kappa\gamma_m\gamma^{(9)}$. Using our methods, we extract a highly non-trivial system of differential and algebraic relations for the associated spinor bilinears, which encodes the geometric constraints imposed on such backgrounds by the requirement that they preserve the stated amount of supersymmetry. For reasons of conceptual and computational convenience, we express such equations in terms of certain combinations of iterated contractions and wedge products which are known as 'generalized products', whose role and origin was explained in [7]. The reader is encouraged at this point to take a look at section 3, which should provide an illustration of the results and techniques developed in the present paper. A full analysis of those equations and of their physical consequences, as well as certain other applications of this formalism, are taken up in subsequent work.

The paper is organized as follows. In section 2, we show how a variant of the cone formalism of [15] and of its cylinder version can be constructed within the geometric algebra approach. In section 3, we apply this formalism to the study of the most general $\mathcal{N} = 2$ compactifications of M-theory on eight-manifolds. We conclude in section 4 with a few remarks on further directions. The physics-oriented reader can start with section 3, before delving into the technical and theoretical details of the other sections. This paper is written as a companion to [7], to which we shall refer repeatedly. Therefore, the reader should have a copy of [7] at hand when approaching the formal developments of section 2.

Notations. As in [7], we let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers, respectively. We work in the smooth differential category, so all manifolds, vector bundles, maps, morphisms of bundles, differential forms etc. are taken to be smooth. We further assume that our connected and smooth manifolds M are paracompact, so that we have partitions of unity subordinate to any open cover. If V is a $\mathbb{K}$-vector bundle over M, we let $\Gamma(M, V)$ denote the space of smooth ($\mathcal{C}^\infty$) sections of V. We also let $\mathrm{End}(V) = \mathrm{Hom}(V, V) \approx V \otimes V^*$ denote the $\mathbb{K}$-vector bundle of endomorphisms of V, where $V^* = \mathrm{Hom}(V, \mathcal{O}_\mathbb{K})$ is the dual vector bundle to V while $\mathcal{O}_\mathbb{K}$ denotes the trivial $\mathbb{K}$-line bundle on M. The unital ring of smooth $\mathbb{K}$-valued functions defined on M is denoted by $\mathcal{C}^\infty(M, \mathbb{K}) = \Gamma(M, \mathcal{O}_\mathbb{K})$. The tensor product of $\mathbb{K}$-vector spaces and $\mathbb{K}$-vector bundles is denoted by $\otimes$, while the tensor product of modules over $\mathcal{C}^\infty(M, \mathbb{K})$ is denoted by

$\otimes_{\mathcal{C}^\infty(M,\mathbb{K})}$; hence $\Gamma(M, V_1 \otimes V_2) \approx \Gamma(M, V_1) \otimes_{\mathcal{C}^\infty(M,\mathbb{K})} \Gamma(M, V_2)$. Setting $T_{\mathbb{K}}M \stackrel{\mathrm{def.}}{=} TM \otimes \mathcal{O}_{\mathbb{K}}$ and $T^*_{\mathbb{K}}M \stackrel{\mathrm{def.}}{=} T^*M \otimes \mathcal{O}_{\mathbb{K}}$, the space of $\mathbb{K}$-valued smooth inhomogeneous globally-defined differential forms on M is denoted by $\Omega_{\mathbb{K}}(M) \stackrel{\mathrm{def.}}{=} \Gamma(M, \wedge T^*_{\mathbb{K}}M)$ and is a $\mathbb{Z}$-graded module over the commutative ring $\mathcal{C}^\infty(M, \mathbb{K})$. The fixed rank components of this graded module are denoted by $\Omega^k_{\mathbb{K}}(M) = \Gamma(M, \wedge^k T^*_{\mathbb{K}}M)$ ($k = 0 \ldots d$, where d is the dimension of M).

The kernel and image of any $\mathbb{K}$-linear map $T : \Gamma(M, V_1) \to \Gamma(M, V_2)$ will be denoted by $\mathcal{K}(T)$ and $\mathcal{I}(T)$; these are $\mathbb{K}$-linear subspaces of $\Gamma(M, V_1)$ and $\Gamma(M, V_2)$, respectively. In the particular case when T is a $\mathcal{C}^\infty(M, \mathbb{K})$-linear map (i.e. when it is a morphism of $\mathcal{C}^\infty(M, \mathbb{K})$-modules), the subspaces $\mathcal{K}(T)$ and $\mathcal{I}(T)$ are $\mathcal{C}^\infty(M, \mathbb{K})$-submodules of $\Gamma(M, V_1)$ and $\Gamma(M, V_2)$, respectively — even in those cases when T is not induced by any bundle morphism from V_1 to V_2. We always denote a morphism $f : V_1 \to V_2$ of $\mathbb{K}$-vector bundles and the $\mathcal{C}^\infty(M, \mathbb{K})$-linear map $\Gamma(M, V_1) \to \Gamma(M, V_2)$ induced by it between the modules of sections by the same symbol. Because of this convention, we clarify that the notations $\mathcal{K}(f) \subset \Gamma(M, V_1)$ and $\mathcal{I}(f) \subset \Gamma(M, V_2)$ denote the kernel and the image of the corresponding map on sections $\Gamma(M, V_1) \xrightarrow{f} \Gamma(M, V_2)$, which in this case are $\mathcal{C}^\infty(M, \mathbb{K})$-submodules of $\Gamma(M, V_1)$ and $\Gamma(M, V_2)$, respectively. In general, there does *not* exist any sub-bundle $\ker f$ of V_1 such that $\mathcal{K}(f) = \Gamma(M, \ker f)$ nor any sub-bundle $\mathrm{im} f$ of V_2 such that $\mathcal{I}(f) = \Gamma(M, \mathrm{im} f)$ — though there exist sheaves $\ker f$ and $\mathrm{im} f$ with the corresponding properties.

Given a pseudo-Riemannian metric g on M of signature (p, q), we let $(e_a)_{a=1\ldots d}$ (where $d = \dim M$) denote a local frame of TM, defined on some open subset U of M. We let e^a be the dual local coframe (= local frame of T^*M), which satisfies $e^a(e_b) = \delta^a_b$ and $\hat{g}(e^a, e^b) = g^{ab}$, where (g^{ab}) is the inverse of the matrix (g_{ab}). The contragradient frame $(e^a)_\#$ and contragradient coframe $(e_a)_\#$ are given by:

$$(e^a)^\# = g^{ab} e_b\,, \quad (e_a)_\# = g_{ab} e^b\,,$$

where the $\#$ subscript and superscript denote the (mutually inverse) musical isomorphisms between $T_{\mathbb{K}}M$ and $T^*_{\mathbb{K}}M$ given respectively by lowering and raising indices with the metric g. We set $e^{a_1\ldots a_k} \stackrel{\mathrm{def.}}{=} e^{a_1} \wedge \ldots \wedge e^{a_k}$ and $e_{a_1\ldots a_k} \stackrel{\mathrm{def.}}{=} e_{a_1} \wedge \ldots \wedge e_{a_k}$ for any $k = 0 \ldots d$. A general $\mathbb{K}$-valued inhomogeneous form $\omega \in \Omega_{\mathbb{K}}(M)$ expands as:

$$\omega = \sum_{k=0}^{d} \omega^{(k)} =_U \sum_{k=0}^{d} \frac{1}{k!} \omega^{(k)}_{a_1\ldots a_k} e^{a_1\ldots a_k}\,, \tag{1.1}$$

where the symbol $=_U$ means that the equality holds only after restriction of ω to U and we have used the expression:

$$\omega^{(k)} =_U \frac{1}{k!} \omega^{(k)}_{a_1\ldots a_k} e^{a_1\ldots a_k}\,. \tag{1.2}$$

The locally-defined smooth functions $\omega^{(k)}_{a_1\ldots a_k} \in \mathcal{C}^\infty(U, \mathbb{K})$ (the 'strict coefficient functions' of ω) are completely antisymmetric in $a_1 \ldots a_k$. Given a pinor bundle on M with underlying fiberwise representation γ of the Clifford bundle of $T^*_{\mathbb{K}}M$, the corresponding gamma 'matrices' in the coframe e^a are denoted by $\gamma^a \stackrel{\mathrm{def.}}{=} \gamma(e^a)$, while the gamma matrices in the

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contragradient coframe $(e_a)_\#$ are denoted by $\gamma_a \stackrel{\text{def.}}{=} \gamma((e_a)_\#) = g_{ab}\gamma^b$. We will occasionally assume that the frame (e_a) is *pseudo-orthonormal* in the sense that e_a satisfy:

$$g(e_a, e_b)\ (= g_{ab})\ = \eta_{ab}\,,$$

where (η_{ab}) is a diagonal matrix with p diagonal entries equal to $+1$ and q diagonal entries equal to -1.

2 The geometric algebra of metric cylinders and cones

In this section, we study the Kähler-Atiyah algebra (see [7] for background) of metric cylinders and cones $(\hat{M}, g_{\rm cyl})$ and $(\hat{M}, g_{\rm cone})$ (figures 1 and 2) over pseudo-Riemannian manifolds (M, g) as well as pin bundles over such spaces, paying special attention to the manner in which constrained generalized Killing pinor equations [7] behave in such cases. Our treatment is motivated by the application considered in section 3, where it is convenient to consider the metric cone or cylinder over a compactification space for reasons related to giving an interpretation through reductions of structure group and intrinsic torsion. Though those aspects of the model considered in section 3 fall largely outside of the scope of the present paper — being, instead, discussed in detail in subsequent work — we encourage the reader to refer to section 3 for one of our motivations for developing the formalism discussed below. We start in subsection 2.1 by recalling some basic facts about the geometry of metric cones and cylinders over a pseudo-Riemannian manifold of even dimension — the case which will form the focus of our considerations, given the application considered in section 3. In subsection 2.2, we discuss the Kähler-Atiyah algebra of metric cones and cylinders, paying special attention to certain subalgebras which play a crucial role in the study of pinors over such spaces. In particular, we consider the subalgebras of twisted (anti-)selfdual forms [7], the subalgebra of so-called *special forms* and the subalgebra of forms which are orthogonal to the (dual of the) generating vector field of the cylinder and cone, respectively. We show that the intersection of the subalgebra of special forms with the subalgebra of forms orthogonal to the generator (an intersection which we call the *subalgebra of vertical forms*) is isomorphic with the Kähler-Atiyah algebra of the unit section (M, g) of the cone or cylinder via a natural geometric isomorphism — a result which allows one to easily lift problems and results from the Kähler-Atiyah algebra of the unit section to the metric cone or cylinder. We also discuss various isomorphic models of this subalgebra. Subsection 2.3 considers — within the geometric algebra framework — the Levi-Civita connections of the cylinder and cone as well as the connections induced by them on the exterior bundle. Subsection 2.4 discusses pin bundles over metric cones and cylinders (in the case when the Clifford algebra associated with the dimension of the cone and with the signature of the cone metric is non-simple — which, once again, is the case relevant for the application considered in section 3). We give an explicit construction of the module structure on such pin bundles, a result which will be useful later. In subsection 2.5, we discuss some basic properties of the Fierz isomorphism [7] of cylinders and cones — in particular, we explain its relation with the Fierz isomorphism of the unit section. In subsections 2.6 and 2.7, we discuss the lift of connections from the pin bundle

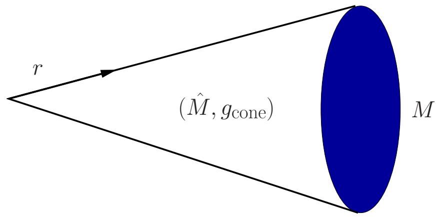

Figure 1. Metric cone over M.

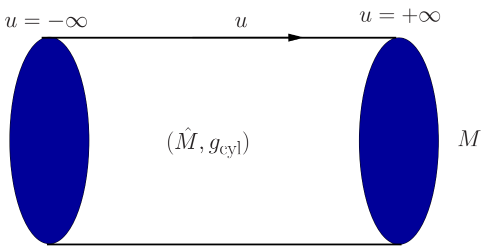

Figure 2. Metric cylinder over M.

of (M, g) to the pin bundle of the cylinder and cone as well as the 'dequantization' of such lifted connections — which (as in [7]) results in certain 'geometric connections' on the Kähler-Atiyah algebra of the cylinder and cone. Subsection 2.8 considers the lift of algebraic constraints on pinors and forms to the metric cylinder and cone while subsection 2.9 discusses the similar lift of generalized Killing conditions on pinors and forms. Subsection 2.10 combines these results to treat the lift of constrained generalized Killing conditions from (M, g) to its metric cylinder and cone — a process which will be used in the example of section 3 in order to simplify the analysis of CGK pinor equations on (M, g). Finally, subsection 2.11 considers certain truncated models which — as in [7] — turn out to be especially amenable to implementation in various symbolic computation systems such as `Ricci` [17] or Cadabra [18].

Simplifying assumptions. Throughout this section, we let (M, g) be a pseudo-Riemannian manifold of *even* dimension d and $\hat{M}$ be a manifold diffeomorphic with $\mathbb{R} \times M$ (on which we shall consider the cylinder and cone metrics, respectively). We let (p, q) be the signature type of the metric g on M; then $\dim \hat{M} = d + 1$ and both the cone and cylinder metric which we shall consider on $\hat{M}$ will have signature type $(p + 1, q)$. We also assume that the Clifford algebra $\mathrm{Cl}_{\mathbb{K}}(p + 1, q)$ is *non-simple* and that the Schur algebra [7] of $\mathrm{Cl}_{\mathbb{K}}(p+1, q)$ equals the base field $\mathbb{K}$, which amounts to assuming that one of the following assumptions hold:

(A) $\mathbb{K} = \mathbb{C}$,

or

(B) $\mathbb{K} = \mathbb{R}$ and $p - q \equiv_8 0$.

With these assumptions, it follows that the Clifford algebra $\mathrm{Cl}_{\mathbb{K}}(p,q)$ which is relevant for (M,g) is simple and that its Schur algebra also equals the base field. We further assume that M is oriented and that on $\hat{M}$ we have chosen the orientation compatible with that of M.

2.1 Preparations

On $\hat{M}$, consider the cylinder metric g_{cyl} whose squared line element takes the form:

$$\mathrm{d}s^2_{\mathrm{cyl}} = \mathrm{d}u^2 + \mathrm{d}s^2 \qquad (u \in \mathbb{R})\,,$$

where $\mathrm{d}s^2$ is the squared line element of g. This is related by a conformal transformation to the cone metric g_{cone} on $\hat{M}$, whose squared line element is given by:

$$\mathrm{d}s^2_{\mathrm{cone}} = \mathrm{d}r^2 + r^2\mathrm{d}s^2 = r^2\mathrm{d}s^2_{\mathrm{cyl}} \qquad (r \stackrel{\mathrm{def.}}{=} e^u \in (0,+\infty))\ .$$

We have $g_{\mathrm{cone}} = r^2 g_{\mathrm{cyl}}$ and[1] $\hat{g}_{\mathrm{cone}} = \frac{1}{r^2}\hat{g}_{\mathrm{cyl}}$, where we view u and $r = e^u$ as smooth functions defined on $\hat{M}$, namely $u \in \mathcal{C}^\infty(\hat{M},\mathbb{R})$ and $r \in \mathcal{C}^\infty(\hat{M},(0,+\infty)) \subset \mathcal{C}^\infty(\hat{M},\mathbb{R})$. The transformation $u \to r$ maps the limit $u \to -\infty$ to the limit $r \to 0$. Unless M is a sphere, the cone metric is not complete due to the conical singularity which arises when one attempts to add the point at $r = 0$. For any vector field $V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$ and any one-form $\eta \in \Gamma(\hat{M}, T^*_{\mathbb{K}}\hat{M}) = \Omega^1_{\mathbb{K}}(\hat{M})$, we have $V_{\#_{\mathrm{cone}}} = r^2 V_{\#_{\mathrm{cyl}}}$ and $\eta^{\#_{\mathrm{cone}}} = \frac{1}{r^2}\eta^{\#_{\mathrm{cyl}}}$, where $\#_{\mathrm{cyl}}$ and $\#_{\mathrm{cone}}$ are the musical isomorphisms of the cylinder and cone, respectively.

The ring $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$. We let $\Pi : \hat{M} \to M$ be the projection on the second factor of the Cartesian product $\hat{M} = \mathbb{R}\times M$. For later reference, consider the following unital subring of the commutative ring $\mathcal{C}^\infty(\hat{M},\mathbb{K})$:

$$\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K}) \stackrel{\mathrm{def.}}{=} \{f\circ\Pi | f \in \mathcal{C}^\infty(M,\mathbb{K})\} \subset \mathcal{C}^\infty(\hat{M},\mathbb{K})\ .$$

It coincides with the image $\Pi^*(\mathcal{C}^\infty(M,\mathbb{K}))$ through the pullback map Π^*, which acts as follows on smooth functions defined on M:

$$\Pi^*(f) = f\circ\Pi \in \mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})\,,\ \ \forall f \in \mathcal{C}^\infty(M,\mathbb{K})\ .$$

In fact, Π^* corestricts to a unital isomorphism of rings:

$$\mathcal{C}^\infty(M,\mathbb{K}) \stackrel{\Pi^*|^{\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})}}{\longrightarrow} \mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})\,,$$

which allows us to identify $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$ with $\mathcal{C}^\infty(M,\mathbb{K})$. The pullback $\Pi^* : \Omega_{\mathbb{K}}(M) \to \Omega_{\mathbb{K}}(\hat{M})$ of $\mathbb{K}$-valued differential forms satisfies:

$$\Pi^*(f\omega) = \Pi^*(f)\Pi^*(\omega)\,,\ \ \forall f \in \mathcal{C}^\infty(M,\mathbb{K})\,,\ \ \forall \omega \in \Omega_{\mathbb{K}}(M,\mathbb{K})$$

and maps wedge products into wedge products. It can therefore be viewed as a morphism of $\mathcal{C}^\infty(M,\mathbb{K})$-algebras from the exterior algebra of M to that of $\hat{M}$, provided that we identify $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$ with $\mathcal{C}^\infty(M,\mathbb{K})$ as explained above.

[1] As usual, $\hat{g}_{\mathrm{cone}}$ and $\hat{g}_{\mathrm{cyl}}$ denote the metrics induced by g_{cone} and g_{cyl} on $T^*_{\mathbb{K}}\hat{M}$.

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The lift of vector fields. Notice that the cone can be viewed as the warped product [19, 20] $(\hat{M}, g_{\rm cone}) \approx ((0,\infty), \mathrm{d}r^2) \times_r (M, \mathrm{d}s^2)$ (of warp factor equal to r) of the positive axis (endowed with the flat metric of squared length element $\mathrm{d}r^2$) with (M,g), while the cylinder is, of course, the direct metric product $(\hat{M}, g_{\rm cyl}) \approx (\mathbb{R}, \mathrm{d}u^2) \times (M, \mathrm{d}s^2)$ of the real axis (endowed with the flat metric of squared length element $\mathrm{d}u^2$) with (M,g). The latter is the same as the warped product $(\mathbb{R}, \mathrm{d}u^2) \times_1 (M,g)$ with constant warp factor equal to one. The pulled-back bundle $\Pi^*(T_{\mathbb{K}}M)$ can be identified with the sub-bundle $T_{\mathbb{K}}^{\perp}\hat{M}$ whose fiber at a point $\hat{x} \in \hat{M}$ is the orthogonal complement in $T_{\mathbb{K},\hat{x}}\hat{M}$ of the tangent vector $(\partial_r)_{\hat{x}} \in T_{\mathbb{K},\hat{x}}\hat{M}$ with respect to $g_{\rm cone}$; of course, this coincides with the orthogonal complement of the vector $(\partial_u)_{\hat{x}}$ with respect to $g_{\rm cyl}$. A vector field $X \in \Gamma(M, T_{\mathbb{K}}M)$ pulls back to the section $\Pi^*(X)$ of the bundle $\Pi^*(T_{\mathbb{K}}M)$, which in turn can be viewed as a section X_* of the sub-bundle $T_{\mathbb{K}}^{\perp}\hat{M} \subset T_{\mathbb{K}}\hat{M}$, i.e. as a vector field on $\hat{M}$ which is everywhere orthogonal to ∂_r (and thus to ∂_u). Of course, X_* coincides with the well-known lift of X along a warped product. Using the identification $\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K}) \approx \mathcal{C}^\infty(M, \mathbb{K})$ discussed above, the pullback of sections can be viewed as a (non-surjective) $\mathcal{C}^\infty(M,\mathbb{K})$-linear map $\Gamma(M, T_{\mathbb{K}}M) \ni X \xrightarrow{\Pi^*} X_* \in \Gamma(M, T_{\mathbb{K}}^{\perp}\hat{M})$.

The canonical normalized one-forms. The one-form:

$$\psi = \mathrm{d}u = \frac{1}{r}\mathrm{d}r$$

has unit norm with respect to the cylinder metric, being dual to the unit norm vector field $\psi^{\#_{\rm cyl}} = \partial_u = r\partial_r$ with respect to the metric $g_{\rm cyl}$:

$$\psi = \partial_u \lrcorner g_{\rm cyl} \ .$$

Similarly, the one-form:

$$\theta = \mathrm{d}r = r\psi$$

has unit norm with respect to the cone metric, being dual to the unit norm vector field $\theta^{\#_{\rm cone}} = \partial_r$ with respect to the metric $g_{\rm cone}$:

$$\theta = \partial_r \lrcorner g_{\rm cone} \ .$$

The pairings (inner products) induced by $g_{\rm cone}$ and $g_{\rm cyl}$ on $\Omega_{\mathbb{K}}(\hat{M})$ are related through:

$$\langle \omega, \eta \rangle_{\rm cone} = \frac{1}{r^{2k}} \langle \omega, \eta \rangle_{\rm cyl} \,, \quad \forall \omega, \eta \in \Omega^k_{\mathbb{K}}(\hat{M}) \ .$$

Together with the definition of the (left) interior product, the last relation implies:

$$\iota^{\rm cone}_\omega = \frac{1}{r^{2k}} \iota^{\rm cyl}_\omega \,, \quad \forall \omega \in \Omega^k_{\mathbb{K}}(\hat{M}) \ . \tag{2.1}$$

In turn, this gives:

$$\iota^{\rm cone}_\theta = \frac{1}{r} \iota^{\rm cyl}_\psi \,,$$

a relation which will be used below. For any vector field V on $\hat{M}$, we let $\mathcal{L}_V$ denote the Lie derivative with respect to V. We have $\mathcal{L}_{fV}\omega = \mathrm{d}f \wedge (V \lrcorner \omega) + f\mathcal{L}_V\omega$ for any $f \in \mathcal{C}^\infty(\hat{M}, \mathbb{K})$ and any $\omega \in \Omega_{\mathbb{K}}(\hat{M})$, a relation which gives:

$$\mathcal{L}_{\partial_u}\omega = r\mathcal{L}_{\partial_r}\omega + \theta \wedge (\partial_r \lrcorner \omega) \,, \quad \forall \omega \in \Omega_{\mathbb{K}}(\hat{M}) \,. \tag{2.2}$$

In turn, this implies:

$$\mathcal{L}_{\partial_r}\omega = \frac{1}{r}(\mathcal{L}_{\partial_u}\omega - \psi \wedge (\partial_u \lrcorner \omega)) \,, \quad \forall \omega \in \Omega_{\mathbb{K}}(\hat{M}) \,, \tag{2.3}$$

where we noticed that $\theta \wedge (\partial_r \lrcorner \omega) = \psi \wedge (\partial_u \lrcorner \omega)$.

The Euler operator. Consider the Euler operator $\mathcal{E} = \oplus_{k=0}^{d+1} k \ \mathrm{id}_{\Omega^k_{\mathbb{K}}(\hat{M})}$ on $\Omega_{\mathbb{K}}(\hat{M})$ associated with the rank decomposition $\Omega_{\mathbb{K}}(\hat{M}) = \oplus_{k=0}^{d+1}\Omega^k_{\mathbb{K}}(\hat{M})$. This acts as follows on a general inhomogeneous form:

$$\mathcal{E}(\omega) = \sum_{k=0}^{d+1} k\omega^{(k)} \,, \quad \forall \omega = \sum_{k=0}^{d+1} \omega^{(k)} \in \Omega_{\mathbb{K}}(\hat{M}) \ \text{ with } \ \omega^{(k)} \in \Omega^k_{\mathbb{K}}(\hat{M}) \,.$$

Notice that $\mathcal{E}$ has eigenvalues $0, \ldots, d+1$, with eigenspaces given by:

$$\mathcal{K}(\mathcal{E} - k \ \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})}) = \Omega^k_{\mathbb{K}}(\hat{M}) \,.$$

Since $\mathcal{E}$ is induced by an endomorphism of the exterior bundle, the exponential $e^{\mathcal{E}} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathcal{E}^{\circ n}$ is well-defined, being itself induced by a bundle endomorphism of $\wedge T^*_{\mathbb{K}}\hat{M}$. We obtain well-defined $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-linear automorphisms $\lambda^{\mathcal{E}} \stackrel{\text{def.}}{=} e^{(\ln \lambda)\mathcal{E}}$ of $\Omega_{\mathbb{K}}(\hat{M})$ for any positive real number $\lambda > 0$ (and similar operators for any non-vanishing complex number λ, provided that we choose a branch of the logarithm). Of course, we have $(\lambda_1\lambda_2)^{\mathcal{E}} = \lambda_1^{\mathcal{E}} \circ \lambda_2^{\mathcal{E}}$ for any $\lambda_1, \lambda_2 > 0$ and $1^{\mathcal{E}} = \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})}$, so the map $(0, +\infty) \ni \lambda \to \lambda^{\mathcal{E}} \in \mathrm{End}_{\mathcal{C}^\infty(\hat{M},\mathbb{K})}(\Omega_{\mathbb{K}}(\hat{M}))$ gives a representation of the multiplicative group of positive reals through $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-linear endomorphisms of $\Omega_{\mathbb{K}}(\hat{M})$. These operators act as:

$$\lambda^{\mathcal{E}}(\omega) = \sum_{k=0}^{d+1} \lambda^k\omega^{(k)} \,, \quad \forall \omega = \sum_{k=0}^{d+1} \omega^{(k)} \in \Omega_{\mathbb{K}}(\hat{M}) \ \text{ with } \ \omega^{(k)} \in \Omega^k_{\mathbb{K}}(\hat{M}) \,.$$

Notice the following relations which hold on $\Omega_{\mathbb{K}}(\hat{M})$:

$$\begin{aligned} r^{\mathcal{E}} \circ \partial_r \lrcorner &= \frac{1}{r}(\partial_r \lrcorner \,) \circ r^{\mathcal{E}}, & r^{\mathcal{E}} \circ \partial_u \lrcorner &= \frac{1}{r}(\partial_u \lrcorner \,) \circ r^{\mathcal{E}} \,, \\ r^{\mathcal{E}} \circ \wedge_\theta &= r \wedge_\theta \circ\, r^{\mathcal{E}}, & r^{\mathcal{E}} \circ \wedge_\psi &= r \wedge_\psi \circ\, r^{\mathcal{E}} \,. \end{aligned} \tag{2.4}$$

They follow from the fact that contraction with a vector field lowers the rank of a homogeneous form by one while wedge product with a one form increases the rank by one. Also notice that the obvious relations:

$$\mathcal{L}_{\partial_r}(r^k) = \frac{k}{r}r^k \Longleftrightarrow \mathcal{L}_{\partial_u}r^k = kr^k \,, \quad \forall k \in \mathbb{Z} \tag{2.5}$$

imply:

$$(\mathcal{L}_{\partial_u} - \mathcal{E}) \circ r^{\mathcal{E}} = r^{\mathcal{E}} \circ \mathcal{L}_{\partial_u} \quad (\text{on } \ \Omega_{\mathbb{K}}(\hat{M})) \,, \tag{2.6}$$

where we used identity (2.3).

Forms parallel and orthogonal to the canonical one-forms. As in [7], let $P_{\|}^{\rm cyl} = \wedge_\psi \circ \iota_\psi^{\rm cyl}$, $P_{\perp}^{\rm cyl} = \iota_\psi^{\rm cyl} \circ \wedge_\psi$ and $P_{\|}^{\rm cone} = \wedge_\theta \circ \iota_\theta^{\rm cone}$, $P_{\perp}^{\rm cone} = \iota_\theta^{\rm cone} \circ \wedge_\theta$ be the projectors on forms parallel and orthogonal to ψ and θ on the cylinder and cone, respectively. Relation (2.1) implies $P_{\|}^{\rm cone} = P_{\|}^{\rm cyl}$ and $P_{\perp}^{\rm cone} = P_{\perp}^{\rm cyl}$, so we define:

$$\begin{aligned} P_{\|} &\stackrel{\rm def.}{=} P_{\|}^{\rm cone} = P_{\|}^{\rm cyl} = \wedge_\psi \circ (\partial_u \lrcorner\,) = \wedge_\theta \circ (\partial_r \lrcorner\,)\,, \\ P_{\perp} &\stackrel{\rm def.}{=} P_{\perp}^{\rm cone} = P_{\perp}^{\rm cyl} = (\partial_u \lrcorner\,) \circ \wedge_\psi = (\partial_r \lrcorner\,) \circ \wedge_\theta\ . \end{aligned}$$

In particular, the decomposition of a form $\omega \in \Omega_{\mathbb{K}}(\hat{M})$ into its part $\omega_{\|} = P_{\|}(\omega)$ parallel to θ (and thus also to ψ) and its part $\omega_\perp = P_\perp(\omega)$ orthogonal to θ (and thus also to ψ) is the same on the cylinder and cone. We let:

$$\begin{aligned} \Omega_{\mathbb{K}}^{\|}(\hat{M}) &\stackrel{\rm def.}{=} P_{\|}(\Omega_{\mathbb{K}}(\hat{M})) = \{\omega \in \Omega_{\mathbb{K}}(\hat{M}) | \omega = \omega_{\|}\}\,, \\ \Omega_{\mathbb{K}}^{\perp}(\hat{M}) &\stackrel{\rm def.}{=} P_{\perp}(\Omega_{\mathbb{K}}(\hat{M})) = \{\omega \in \Omega_{\mathbb{K}}(\hat{M}) | \omega = \omega_{\perp}\}\,, \end{aligned}$$

obtaining the same decomposition $\Omega_{\mathbb{K}}(\hat{M}) = \Omega_{\mathbb{K}}^{\|}(\hat{M}) \oplus \Omega_{\mathbb{K}}^{\perp}(\hat{M})$ for the cylinder and cone. Notice the natural isomorphism $\Omega_{\mathbb{K}}^{\perp}(\hat{M}) \approx \Gamma(\hat{M}, \wedge(T_{\mathbb{K}}^{\perp}\hat{M})^*)$, which we shall often use tacitly later on. Relations (2.4) imply:

$$[r^{\mathcal{E}}, P_{\|}]_{-,\circ} = [r^{\mathcal{E}}, P_{\perp}]_{-,\circ} = 0 \Longleftrightarrow r^{\mathcal{E}}(\omega)_{\|} = r^{\mathcal{E}}(\omega_{\|}) \ \text{ and } \ r^{\mathcal{E}}(\omega)_{\perp} = r^{\mathcal{E}}(\omega_{\perp})\ .$$

In particular, we have:

$$r^{\mathcal{E}}(\Omega_{\mathbb{K}}^{\|}(\hat{M})) = \Omega_{\mathbb{K}}^{\|}(\hat{M})\,, \quad r^{\mathcal{E}}(\Omega_{\mathbb{K}}^{\perp}(\hat{M})) = \Omega_{\mathbb{K}}^{\perp}(\hat{M})\ . \tag{2.7}$$

On the other hand, the relation $\partial_r \lrcorner \theta = 1$ and the fact that θ is closed imply $\mathcal{L}_{\partial_r}\theta = 0$ upon using the identity $\mathcal{L}_V \omega = \mathrm{d}(V \lrcorner \omega) + V \lrcorner (\mathrm{d}\omega)$, which holds for any vector field V and any inhomogeneous form ω defined on $\hat{M}$. In turn, this implies that $\mathcal{L}_{\partial_r}$ commutes with the operator $\wedge_\theta$ and (since it commutes with the operator $\partial_r \lrcorner$) also with the projectors $P_{\|}$ and $P_\perp$:

$$[\mathcal{L}_{\partial_r}, \partial_r \lrcorner\,]_{-,\circ} = [\mathcal{L}_{\partial_r}, \wedge_\theta]_{-,\circ} = 0 \ \Longrightarrow [\mathcal{L}_{\partial_r}, P_{\|}]_{-,\circ} = [\mathcal{L}_{\partial_r}, P_{\perp}]_{-,\circ} = 0\ .$$

Similarly, we have the commutation relations:

$$[\mathcal{L}_{\partial_u}, \partial_u \lrcorner\,]_{-,\circ} = [\mathcal{L}_{\partial_u}, \wedge_\psi]_{-,\circ} = 0 \ \Longrightarrow [\mathcal{L}_{\partial_u}, P_{\|}]_{-,\circ} = [\mathcal{L}_{\partial_u}, P_{\perp}]_{-,\circ} = 0\,,$$

as a consequence of the identity $\partial_u \lrcorner \psi = 1$, which implies $\mathcal{L}_{\partial u}\psi = 0$. Also notice that relation (2.2) reads:

$$\mathcal{L}_{\partial_u} = r\mathcal{L}_{\partial_r} + P_{\|}\ .$$

The commutation relations given above imply:

$$\begin{aligned} &(\mathcal{L}_{\partial_r}\omega)_{\|} = \mathcal{L}_{\partial_r}(\omega_{\|})\,, && (\mathcal{L}_{\partial_r}\omega)_{\perp} = \mathcal{L}_{\partial_r}(\omega_{\perp})\,, \\ &(\mathcal{L}_{\partial_u}\omega)_{\|} = \mathcal{L}_{\partial_u}(\omega_{\|})\,, && (\mathcal{L}_{\partial_u}\omega)_{\perp} = \mathcal{L}_{\partial_u}(\omega_{\perp})\,, \end{aligned} \tag{2.8}$$

for all $\omega \in \Omega_{\mathbb{K}}(\hat{M})$. In particular, we have:

$$\begin{aligned} &\mathcal{L}_{\partial_r}(\Omega_{\mathbb{K}}^{\perp}(\hat{M})) \subset \Omega_{\mathbb{K}}^{\perp}(\hat{M})\,, && \mathcal{L}_{\partial_r}(\Omega_{\mathbb{K}}^{\|}(\hat{M})) \subset \Omega_{\mathbb{K}}^{\|}(\hat{M})\,, \\ &\mathcal{L}_{\partial_u}(\Omega_{\mathbb{K}}^{\perp}(\hat{M})) \subset \Omega_{\mathbb{K}}^{\perp}(\hat{M})\,, && \mathcal{L}_{\partial_u}(\Omega_{\mathbb{K}}^{\|}(\hat{M})) \subset \Omega_{\mathbb{K}}^{\|}(\hat{M})\ . \end{aligned}$$

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(Conformal) Killing properties. Finally, note that ∂_r is a normalized conformal Killing vector field for $g_{\rm cone}$ and a Killing vector field for $g_{\rm cyl}$, while $\partial_u = r\partial_r$ is a homothety for $g_{\rm cone}$ and a normalized Killing vector field for $g_{\rm cyl}$:

$$\begin{aligned} &\mathcal{L}_{\partial_r} g_{\rm cone} = \frac{2}{r} g_{\rm cone}, && \mathcal{L}_{\partial_r} g_{\rm cyl} = 0\,, \\ &\mathcal{L}_{\partial_u} g_{\rm cone} = 2 g_{\rm cone}, && \mathcal{L}_{\partial_u} g_{\rm cyl} = 0\ . \end{aligned} \tag{2.9}$$

Thus θ and ψ are Killing-Yano one-forms (of various types) with respect to both metrics.

2.2 The Kähler-Atiyah algebra of metric cones and cylinders over pseudo-Riemannian manifolds

Relating the Kähler-Atiyah algebras of the cylinder and cone. We let $\diamond$ and $\triangle_p$ ($p = 0 \ldots d$) denote the geometric and generalized products constructed on $\Omega_{\mathbb{K}}(M)$ using the metric g. Similarly, we let $\diamond^{\rm cyl}$, $\triangle_p^{\rm cyl}$ and $\diamond^{\rm cone}$, $\triangle_p^{\rm cone}$ ($p = 0 \ldots d+1$) denote the geometric and generalized products on $\Omega_{\mathbb{K}}(\hat{M})$ induced by the metrics $g_{\rm cyl}$ and $g_{\rm cone}$ respectively. Using the definition of generalized products, we find:

$$\triangle_p^{\rm cone} = \frac{1}{r^{2p}} \triangle_p^{\rm cyl}\,, \quad \forall p = 0 \ldots d+1\ . \tag{2.10}$$

Since the generalized product $\triangle_p$ is homogeneous of degree $-2p$ when viewed as a map $\triangle_p : \Omega_{\mathbb{K}}(\hat{M}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \Omega_{\mathbb{K}}(\hat{M}) \to \Omega_{\mathbb{K}}(\hat{M})$ from the tensor product $\Omega_{\mathbb{K}}(\hat{M}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \Omega_{\mathbb{K}}(\hat{M})$ (endowed with the grading induced by the rank grading of the exterior algebra) to $\Omega_{\mathbb{K}}(\hat{M})$, the following identities hold for all $p = 0 \ldots d+1$:

$$\begin{aligned} (\mathcal{E} + 2p\ \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})}) \circ \triangle_p^{\rm cyl} &= \triangle_p^{\rm cyl} \circ (\mathcal{E} \otimes \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} + \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} \otimes \mathcal{E})\,, \\ (\mathcal{E} + 2p\ \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})}) \circ \triangle_p^{\rm cone} &= \triangle_p^{\rm cone} \circ (\mathcal{E} \otimes \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} + \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} \otimes \mathcal{E})\,, \end{aligned} \tag{2.11}$$

i.e.:

$$\begin{aligned} (\mathcal{E} + 2p\ \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})})(\omega \triangle_p^{\rm cyl} \eta) &= \mathcal{E}(\omega) \triangle_p^{\rm cyl} \eta + \omega \triangle_p^{\rm cyl} \mathcal{E}(\eta)\,, \quad \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})\,, \\ (\mathcal{E} + 2p\ \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})})(\omega \triangle_p^{\rm cone} \eta) &= \mathcal{E}(\omega) \triangle_p^{\rm cone} \eta + \omega \triangle_p^{\rm cone} \mathcal{E}(\eta)\,, \quad \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})\ . \end{aligned}$$

These identities imply:

$$\begin{aligned} r^{\mathcal{E}} \circ \triangle_p^{\rm cyl} &= \frac{1}{r^{2p}}\ \triangle_p^{\rm cyl} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) \Longleftrightarrow \\ r^{\mathcal{E}}(\omega \triangle_p^{\rm cyl} \eta) &= \frac{1}{r^{2p}} [r^{\mathcal{E}}(\omega) \triangle_p^{\rm cyl} r^{\mathcal{E}}(\eta)]\,, \quad \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})\,, \\ r^{\mathcal{E}} \circ \triangle_p^{\rm cone} &= \frac{1}{r^{2p}}\ \triangle_p^{\rm cone} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) \Longleftrightarrow \\ r^{\mathcal{E}}(\omega \triangle_p^{\rm cone} \eta) &= \frac{1}{r^{2p}} [r^{\mathcal{E}}(\omega) \triangle_p^{\rm cone} r^{\mathcal{E}}(\eta)]\,, \ \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})\ . \end{aligned}$$

Combining the first of these relations with (2.10) gives:

$$r^{\mathcal{E}} \circ \triangle_p^{\rm cyl} = \triangle_p^{\rm cone} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) \Longleftrightarrow r^{\mathcal{E}}(\omega \triangle_p^{\rm cyl} \eta) = r^{\mathcal{E}}(\omega) \triangle_p^{\rm cone} r^{\mathcal{E}}(\eta)\,, \quad \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})\,,$$

which in turn implies:

$$r^{\mathcal{E}} \circ \diamond^{\rm cyl} = \diamond^{\rm cone} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) \Longleftrightarrow r^{\mathcal{E}}(\omega \diamond^{\rm cyl} \eta) = r^{\mathcal{E}}(\omega) \diamond^{\rm cone} r^{\mathcal{E}}(\eta)\,, \quad \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})\,. \tag{2.12}$$

Together with the obvious relation $r^{\mathcal{E}}(1_{\hat{M}}) = 1_{\hat{M}}$ (which follows from $\mathcal{E}(1_{\hat{M}}) = 0$), this shows that the Kähler-Atiyah algebras of the cylinder and cone can be identified through appropriate rescalings of their fixed rank subspaces:

Proposition. The maps $r^{\mathcal{E}}$ and $r^{-\mathcal{E}}$ are mutually inverse $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-linear unital isomorphisms of algebras between the Kähler-Atiyah algebras of the cylinder and cone:

$$(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl}) \;\underset{r^{-\mathcal{E}}}{\overset{r^{\mathcal{E}}}{\rightleftarrows}}\; (\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone})\,. \tag{2.13}$$

For later reference, we note the following identities which are an easy consequence of the previous proposition:

$$L^{\rm cone}_{r^{\mathcal{E}}(\omega)} = r^{\mathcal{E}} \circ L^{\rm cyl}_{\omega} \circ r^{-\mathcal{E}}\,, \quad R^{\rm cone}_{r^{\mathcal{E}}(\omega)} = r^{\mathcal{E}} \circ R^{\rm cyl}_{\omega} \circ r^{-\mathcal{E}}\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(\hat{M})\,, \tag{2.14}$$

where $L^{\rm cyl}_\omega, R^{\rm cyl}_\omega$ and $L^{\rm cone}_\omega, R^{\rm cone}_\omega$ are the operators of left and right multiplication with ω in the Kähler-Atiyah algebras of the cylinder and cone.

Relating $(\Omega^{\perp}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl})$ and $(\Omega^{\perp}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone})$. Recalling properties (2.7) and the fact that $\Omega^{\perp}_{\mathbb{K}}(\hat{M})$ is a unital subalgebra of the Kähler-Atiyah algebras of the cylinder and cone (see [7]), the previous proposition implies:

Corollary. The maps $r^{\mathcal{E}}$ and $r^{-\mathcal{E}}$ restrict to mutually inverse unital isomorphisms between the algebras $(\Omega^{\perp}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl})$ and $(\Omega^{\perp}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone})$:

$$(\Omega^{\perp}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl}) \;\underset{r^{-\mathcal{E}}|_{\Omega^{\perp}_{\mathbb{K}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\perp}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}}\; (\Omega^{\perp}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone})\,. \tag{2.15}$$

The special and vertical subalgebras. Using the fact that $\mathcal{L}_{\partial_u}$ is a degree zero $\mathbb{K}$-linear derivation of the exterior algebra, the last of relations (2.9) and the identity $[\mathcal{L}_{\partial_u}, \partial_u \lrcorner]_{\circ,-} = 0$ imply that $\mathcal{L}_{\partial_u}$ is a $\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K})$-linear derivation of degree zero of all generalized products of the cylinder:

$$\mathcal{L}_{\partial_u} \circ \triangle^{\rm cyl}_p = \triangle^{\rm cyl}_p \circ (\mathcal{L}_{\partial_u} \otimes \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} + \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} \otimes \mathcal{L}_{\partial_u})\,,$$

a relation which encodes the fact that $\triangle_p$ are invariant under the translations $u \to u + \epsilon$ ($\epsilon \in \mathbb{R}$) along the generator of the cylinder. This implies that $\mathcal{L}_{\partial_u}$ is an even $\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K})$-linear derivation of the Kähler-Atiyah algebra $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl})$:

$$\mathcal{L}_{\partial_u} \circ \diamond^{\rm cyl} = \diamond^{\rm cyl} \circ (\mathcal{L}_{\partial_u} \otimes \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} + \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} \otimes \mathcal{L}_{\partial_u})\,. \tag{2.16}$$

Using (2.10) and (2.5), the relations given above imply:

$$(\mathcal{L}_{\partial_u} + 2p\,\mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})}) \circ \triangle_p^{\mathrm{cone}} = \triangle_p^{\mathrm{cone}} \circ (\mathcal{L}_{\partial_u} \otimes \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} + \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} \otimes \mathcal{L}_{\partial_u})\ .$$

Combining this with (2.11) shows that the operator $\mathcal{L}_{\partial_u} - \mathcal{E}$ is a degree zero $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear derivation of all generalized products of the cone:

$$(\mathcal{L}_{\partial_u} - \mathcal{E}) \circ \triangle_p^{\mathrm{cone}} = \triangle_p^{\mathrm{cone}} \circ \Big[(\mathcal{L}_{\partial_u} - \mathcal{E}) \otimes \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} + \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} \otimes (\mathcal{L}_{\partial_u} - \mathcal{E})\Big]$$

and hence this operator is an even $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear derivation of the Kähler-Atiyah algebra $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}})$:

$$(\mathcal{L}_{\partial_u} - \mathcal{E}) \circ \diamond^{\mathrm{cone}} = \diamond^{\mathrm{cone}} \circ \Big[(\mathcal{L}_{\partial_u} - \mathcal{E}) \otimes \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} + \mathrm{id}_{\Omega_{\mathbb{K}}(\hat{M})} \otimes (\mathcal{L}_{\partial_u} - \mathcal{E})\Big]\ . \tag{2.17}$$

In particular, we have:

$$(\mathcal{L}_{\partial_u} - \mathcal{E})(\omega \diamond^{\mathrm{cone}} \eta) = [(\mathcal{L}_{\partial_u} - \mathcal{E})\omega] \diamond^{\mathrm{cone}} \eta + \omega \diamond^{\mathrm{cone}} [(\mathcal{L}_{\partial_u} - \mathcal{E})\eta]\ ,\quad \forall \omega, \eta \in \Omega_{\mathbb{K}}(\hat{M})\ .$$

Notice that $\mathcal{L}_{\partial_u}$ is *not* a derivation of the Kähler-Atiyah algebra of the cone. The observations above imply that the following subspaces of $\Omega_{\mathbb{K}}(\hat{M})$:

$$\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \mathcal{K}(\mathcal{L}_{\partial_u})\ ,\quad \Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \mathcal{K}\left(\mathcal{L}_{\partial_u} - \mathcal{E}\right)$$

are unital $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebras of the Kähler-Atiyah algebras $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}})$ and $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}})$, which we shall call the *special subalgebras* of the cylinder and cone, respectively. The previous proposition and relation (2.6) give:

Proposition. The appropriate restrictions of the maps $r^{\pm\mathcal{E}}$ give mutually inverse $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras between the special subalgebras of the cylinder and cone:

$$(\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}}) \quad \underset{r^{-\mathcal{E}}|_{\Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}} \quad (\Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}})\ . \tag{2.18}$$

We also know from [7] that the subspace:

$$\Omega^{\perp}_{\mathbb{K}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \{\omega \in \Omega_{\mathbb{K}}(\hat{M}) | \partial_u \lrcorner \omega = 0\} = \{\omega \in \Omega_{\mathbb{K}}(\hat{M}) | \partial_r \lrcorner \omega = 0\}$$

is a unital $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-subalgebra of both $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}})$ and $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}})$. Therefore, the intersections:

$$\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \Omega^{\perp}_{\mathbb{K}}(\hat{M}) \cap \Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M})\ ,\quad \Omega^{\perp,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \Omega^{\perp}_{\mathbb{K}}(\hat{M}) \cap \Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M})$$

are unital $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebras $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}})$ and $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}})$ respectively, which we shall call the *vertical subalgebras* of the cylinder and cone. Since $\partial_r \lrcorner\, |_{\Omega^{\perp}_{\mathbb{K}}(\hat{M})} = 0$, equations (2.2) and (2.6) give:

$$\left(\mathcal{L}_{\partial_r} - \frac{\mathcal{E}}{r}\right) \circ r^{\mathcal{E}}|_{\Omega^{\perp}_{\mathbb{K}}(\hat{M})} = \frac{1}{r} r^{\mathcal{E}} \circ \mathcal{L}_{\partial_u}|_{\Omega^{\perp}_{\mathbb{K}}(\hat{M})}\ . \tag{2.19}$$

The observations made above imply that the operator $r^{\mathcal{E}}$ maps the vertical subalgebra of the cylinder into that of the cone:

$$r^{\mathcal{E}}(\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})) = \Omega^{\perp,\mathrm{cone}}_{\mathbb{K}}(\hat{M})\ .$$

Combining this with the previous proposition, we find:

Proposition. The appropriate restrictions of the maps $r^{\pm\mathcal{E}}$ give mutually inverse $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras between the vertical subalgebras of the cylinder and cone:

$$(\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}}) \quad \underset{r^{-\mathcal{E}}|_{\Omega^{\perp,\mathrm{cone}}_{\mathbb{K}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}} \quad (\Omega^{\perp,\mathrm{cone}}_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}})\quad . \tag{2.20}$$

The modified volume forms of the cylinder and of the cone. Since $\mathrm{Cl}_{\mathbb{K}}(p+1,q)$ is assumed to be non-simple, the volume forms ν^{cone} and ν^{cyl} defined by g_{cone} and g_{cyl} on $\hat{M}$ square to $1_{\hat{M}}$ and are central in $(\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cone}})$ and in $(\Omega_{\mathbb{K}}(\hat{M}),\diamond^{\mathrm{cyl}})$, respectively; they are given explicitly by:

$$\nu^{\mathrm{cyl}} = \psi\wedge\nu^{\mathrm{cyl}}_{\top} = \psi\diamond^{\mathrm{cyl}}\nu^{\mathrm{cyl}}_{\top}\,,\quad \nu^{\mathrm{cone}} = \theta\wedge\nu^{\mathrm{cone}}_{\top} = \theta\diamond^{\mathrm{cone}}\nu^{\mathrm{cone}}_{\top} = r^{d+1}\nu^{\mathrm{cyl}} = r^{\mathcal{E}}(\nu^{\mathrm{cyl}})\ . \tag{2.21}$$

Here:

$$\begin{aligned}\nu^{\mathrm{cyl}}_{\top} &= \iota^{\mathrm{cyl}}_{\psi}\nu_{\mathrm{cyl}} = \partial_u\lrcorner\nu^{\mathrm{cyl}} = \Pi^*(\nu)\,,\ \ \nu^{\mathrm{cone}}_{\top} = \iota^{\mathrm{cone}}_{\theta}\nu^{\mathrm{cone}} = \partial_r\lrcorner\nu^{\mathrm{cone}}\\ &= r^d\Pi^*(\nu) = r^d\nu^{\mathrm{cyl}}_{\top} = r^{\mathcal{E}}(\nu^{\mathrm{cyl}}_{\top})\,,\end{aligned} \tag{2.22}$$

where ν is the volume form of (M,g).

Special twisted (anti-)selfdual forms on cylinders and cones. The last relation in (2.21) and the second relation in (2.14) imply that the twisted Hodge operators $\tilde{*}_{\mathrm{cyl}} = R^{\mathrm{cyl}}_{\nu^{\mathrm{cyl}}}$ of the cylinder and $\tilde{*}_{\mathrm{cone}} = R^{\mathrm{cone}}_{\nu^{\mathrm{cone}}}$ of the cone are related through:

$$\tilde{*}_{\mathrm{cone}}\circ r^{\mathcal{E}} = r^{\mathcal{E}}\circ\tilde{*}_{\mathrm{cyl}} \Longleftrightarrow P^{\mathrm{cone}}_{\pm}\circ r^{\mathcal{E}} = r^{\mathcal{E}}\circ P^{\mathrm{cyl}}_{\pm}\ . \tag{2.23}$$

This implies that $r^{\mathcal{E}}$ identifies the subalgebras:

$$\begin{aligned}\Omega^{\pm}_{\mathbb{K},\mathrm{cyl}}(\hat{M}) &\stackrel{\mathrm{def.}}{=} P^{\mathrm{cyl}}_{\pm}(\Omega_{\mathbb{K}}(\hat{M})) = \{\omega\in\Omega_{\mathbb{K}}(\hat{M})|\tilde{*}_{\mathrm{cyl}}\omega = \pm\omega\}\,,\\ \Omega^{\pm}_{\mathbb{K},\mathrm{cone}}(\hat{M}) &\stackrel{\mathrm{def.}}{=} P^{\mathrm{cone}}_{\pm}(\Omega_{\mathbb{K}}(\hat{M})) = \{\omega\in\Omega_{\mathbb{K}}(\hat{M})|\tilde{*}_{\mathrm{cone}}\omega = \pm\omega\end{aligned}$$

of twisted (anti-)selfdual forms on the cylinder and cone:

$$r^{\mathcal{E}}(\Omega^{\pm}_{\mathbb{K},\mathrm{cyl}}(\hat{M})) = \Omega^{\pm}_{\mathbb{K},\mathrm{cone}}(\hat{M})\ .$$

In fact, we have mutually inverse $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras:

$$(\Omega^{\pm}_{\mathbb{K},\mathrm{cyl}}(\hat{M}),\diamond^{\mathrm{cyl}}) \quad \underset{r^{-\mathcal{E}}|_{\Omega^{\pm}_{\mathbb{K},\mathrm{cone}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\pm}_{\mathbb{K},\mathrm{cyl}}(\hat{M})}}{\rightleftarrows}} \quad (\Omega^{\pm}_{\mathbb{K},\mathrm{cone}}(\hat{M}),\diamond^{\mathrm{cone}})\quad , \tag{2.24}$$

where unitality follows from (2.21), which implies $r^{\mathcal{E}}(p_\pm^{\rm cyl})=p_\pm^{\rm cone}$ (recall from [7] that $p_\pm^{\rm cyl}=\frac{1}{2}(1\pm\nu^{\rm cyl})$ and $p_\pm^{\rm cone}=\frac{1}{2}(1\pm\nu^{\rm cone})$ are the unit elements of the algebras $(\Omega^\pm_{\mathbb{K},{\rm cyl}}(\hat{M}),\diamond^{\rm cyl})$ and $(\Omega^\pm_{\mathbb{K},{\rm cone}}(\hat{M}),\diamond^{\rm cone})$, since $d+1$ is odd and thus $\nu^{\rm cyl},\nu^{\rm cone}$ are central in the corresponding Kähler-Atiyah algebras).

On the other hand, relations (2.9) imply:

$$\mathcal{L}_{\partial_u}\nu^{\rm cyl}=0\,,$$

and (since $\nu^{\rm cone}=r^{d+1}\nu^{\rm cyl}=r^{\mathcal{E}}(\nu^{\rm cyl})$):

$$(\mathcal{L}_{\partial_u}-\mathcal{E})\nu^{\rm cone}=(\mathcal{L}_{\partial_u}-(d+1))\nu^{\rm cone}=0\,.$$

In particular, we have $\nu^{\rm cyl}\in\Omega^{\rm cyl}_{\mathbb{K}}(\hat{M})$ and $\nu^{\rm cone}\in\Omega^{\rm cone}_{\mathbb{K}}(\hat{M})$. Since $\tilde{*}_{\rm cyl}(\omega)=\omega\diamond^{\rm cyl}\nu^{\rm cyl}$ and $\tilde{*}_{\rm cone}(\omega)=\omega\diamond^{\rm cone}\nu^{\rm cone}$ for all $\omega\in\Omega_{\mathbb{K}}(\hat{M})$, the properties listed above imply:

$$[\mathcal{L}_{\partial_u},\tilde{*}_{\rm cyl}]_{-,\circ}=0\Longleftrightarrow[\mathcal{L}_{\partial_u},P_\pm^{\rm cyl}]_{-,\circ}=0\,,\quad[\mathcal{L}_{\partial_u}-\mathcal{E},\tilde{*}_{\rm cone}]_{-,\circ}=0\Longleftrightarrow[\mathcal{L}_{\partial_u}-\mathcal{E},P_\pm^{\rm cone}]_{-,\circ}=0\,,\tag{2.25}$$

where we used the fact that $\mathcal{L}_{\partial_u}$ and $\mathcal{L}_{\partial_u}-\mathcal{E}$ are even derivations of the Kähler-Atiyah algebras of the cylinder and cone, respectively. In particular, the operators $\mathcal{L}_{\partial_u}$ and $\mathcal{L}_{\partial_u}-\mathcal{E}$ preserve the subspaces of twisted (anti-)selfdual forms on the cylinder and cone, respectively:

$$\mathcal{L}_{\partial_u}(\Omega^\pm_{\mathbb{K},{\rm cyl}}(\hat{M}))\subset\Omega^\pm_{\mathbb{K},{\rm cyl}}(\hat{M})\,,\quad(\mathcal{L}_{\partial_u}-\mathcal{E})(\Omega^\pm_{\mathbb{K},{\rm cone}}(\hat{M}))\subset\Omega^\pm_{\mathbb{K},{\rm cone}}(\hat{M})$$

Definition. The subalgebras of *special twisted (anti-)selfdual forms* are the following $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebras of the Kähler-Atiyah algebras of the cylinder and of the cone:

$$\Omega^{\pm,{\rm cyl}}_{\mathbb{K}}(\hat{M})\stackrel{\rm def.}{=}\Omega^\pm_{\mathbb{K},{\rm cyl}}(\hat{M})\cap\Omega^{\rm cyl}_{\mathbb{K}}(\hat{M})\,,\quad\Omega^{\pm,{\rm cone}}_{\mathbb{K}}(\hat{M})\stackrel{\rm def.}{=}\Omega^\pm_{\mathbb{K},{\rm cone}}(\hat{M})\cap\Omega^{\rm cone}_{\mathbb{K}}(\hat{M})\,.$$

These algebras have units $p_\pm^{\rm cyl}=\frac{1}{2}(1\pm\nu^{\rm cyl})$ and $p_\pm^{\rm cone}=\frac{1}{2}(1\pm\nu^{\rm cone})$, respectively. Combining the observations above gives:

Proposition. The appropriate restrictions of the maps $r^{\pm\mathcal{E}}$ give mutually inverse $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-linear unital isomorphisms of algebras between the subalgebras of special twisted selfdual/anti-selfdual forms of the cylinder and cone:

$$(\Omega^{\pm,{\rm cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\rm cyl})\ \underset{r^{-\mathcal{E}}|_{\Omega^{\pm,{\rm cone}}_{\mathbb{K}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega^{\pm,{\rm cyl}}_{\mathbb{K}}(\hat{M})}}{\rightleftarrows}}\ (\Omega^{\pm,{\rm cone}}_{\mathbb{K}}(\hat{M}),\diamond^{\rm cone}).\tag{2.26}$$

Recovering the Kähler-Atiyah algebra of (M,g). The $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebras $(\Omega^{\perp,{\rm cyl}}_{\mathbb{K}}(\hat{M}),\diamond^{\rm cyl})$ and $(\Omega^{\perp,{\rm cone}}_{\mathbb{K}}(\hat{M}),\diamond^{\rm cone})$ can be identified with the Kähler-Atiyah algebra $(\Omega_{\mathbb{K}}(M),\diamond)$ as follows. Let $\Pi:\hat{M}\to M$ be the projection on the second factor. Then one has the following quite obvious statement:

Proposition. The pullback map $\Pi^*: \Omega_{\mathbb{K}}(M) \to \Omega_{\mathbb{K}}(\hat{M})$ has image equal to $\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})$. Furthermore, its corestriction to this image (which we again denote by Π^*) is a unital $\mathcal{C}^\infty(M,\mathbb{K})$-linear isomorphism of algebras from $(\Omega_{\mathbb{K}}(M),\diamond)$ to the vertical subalgebra $(\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M}),\diamond^{\mathrm{cyl}})$ of the cylinder, provided that we identify $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K}) \approx \mathcal{C}^\infty(M,\mathbb{K})$. The inverse of this isomorphism is the pullback map j^*, where $j: M \hookrightarrow \hat{M}$ is the embedding of M as the section $r=1$ of $\hat{M}$. Thus, we have mutually inverse unital isomorphisms of $\mathcal{C}^\infty(M,\mathbb{K}) \approx \mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebras:

$$(\Omega_{\mathbb{K}}(M),\diamond) \; \underset{j^*|_{\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})}}{\overset{\Pi^*|^{\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})}}{\rightleftarrows}} \; (\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M}),\diamond^{\mathrm{cyl}})\,. \tag{2.27}$$

The proof is easy and left to the reader. Combining with the previous proposition gives the relation between the Kähler-Atiyah algebra of M and the vertical subalgebra of the cone:

Proposition. We have mutually-inverse unital isomorphisms of $\mathbb{K}$-algebras:

$$(\Omega_{\mathbb{K}}(M),\diamond) \; \underset{j^*\circ\, r^{-\mathcal{E}}|_{\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M})}}{\overset{r^{\mathcal{E}}\circ\, \Pi^*|^{\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M})}}{\rightleftarrows}} \; (\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M}),\diamond^{\mathrm{cone}})\,. \tag{2.28}$$

Thus $\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})$ consists of those inhomogeneous forms on $\hat{M}$ which are Π-pullbacks of inhomogeneous forms ω on M; this pullback will be called the *cylinder lift* ω_{cyl} of ω:

$$\omega_{\mathrm{cyl}} \overset{\mathrm{def.}}{=} \Pi^*(\omega) \in \Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(M)\;. \tag{2.29}$$

On the other hand, $\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M})$ consists of forms obtained by rescaling the various rank components of such pullbacks with appropriate non-negative powers of r. Explicitly, $\Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(M)$ consists of *cone lifts*:

$$\omega_{\mathrm{cone}} \overset{\mathrm{def.}}{=} r^{\mathcal{E}}(\Pi^*(\omega)) \in \Omega_{\mathbb{K}}^{\perp,\mathrm{cone}}(\hat{M})\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(M)\,, \tag{2.30}$$

which are inhomogeneous forms of the type:

$$\omega_{\mathrm{cone}} = r^{\mathcal{E}}(\Pi^*(\omega)) = \sum_{k=0}^{d} r^k \Pi^*(\omega^{(k)})\,, \quad \forall \omega = \sum_{k=0}^{d} \omega^{(k)} \in \Omega_{\mathbb{K}}(M) \;\;\text{with}\;\; \omega^{(k)} \in \Omega^k_{\mathbb{K}}(M)\;.$$

For example, relations (2.22) show that $\nu_\top^{\mathrm{cyl}}, \nu_\top^{\mathrm{cone}}$ are the cylinder and cone lifts of ν:

$$\nu_\top^{\mathrm{cyl}} = \nu_{\mathrm{cyl}}\,, \quad \nu_\top^{\mathrm{cone}} = \nu_{\mathrm{cone}}\;.$$

Truncated models. As in [7], we consider the complementary idempotent operators $P_<: \Omega_{\mathbb{K}}(\hat{M}) \to \Omega^<_{\mathbb{K}}(\hat{M})$ and $P_>: \Omega_{\mathbb{K}}(\hat{M}) \to \Omega^>_{\mathbb{K}}(\hat{M})$ which associate to an inhomogeneous form the sum of its components of rank smaller, respectively bigger than the half-integer number $\frac{1}{2}\dim \hat{M} = \frac{d+1}{2}$. Since both $r^{\mathcal{E}}$ and the Lie derivative with respect to a vector field preserve the rank of differential forms, we have the commutation relations:

$$[r^{\mathcal{E}}, P_<]_{-,\circ} = [r^{\mathcal{E}}, P_>]_{-,\circ} = [\mathcal{L}_{\partial_u}, P_<]_{-,\circ} = [\mathcal{L}_{\partial_u}, P_>]_{-,\circ} = [\mathcal{L}_{\partial_u} - \mathcal{E}, P_<]_{-,\circ} = [\mathcal{L}_{\partial_u} - \mathcal{E}, P_>]_{-,\circ} = 0 \tag{2.31}$$

In particular, these give:

$$r^{\mathcal{E}}(\Omega_{\mathbb{K}}^{<}(\hat{M})) = \Omega_{\mathbb{K}}^{<}(\hat{M}), \quad \mathcal{L}_{\partial_u}(\Omega_{\mathbb{K}}^{<}(\hat{M})) \subset \Omega_{\mathbb{K}}^{<}(\hat{M}), \quad (\mathcal{L}_{\partial_r} - \mathcal{E})(\Omega_{\mathbb{K}}^{<}(\hat{M})) \subset \Omega_{\mathbb{K}}^{<}(\hat{M}) .$$

Recall from [7] that the subspace $\Omega_{\mathbb{K}}^{<}(\hat{M})$ carries associative binary products defined through:

$$\blacklozenge_{\pm}^{\mathrm{cyl}} \stackrel{\mathrm{def.}}{=} 2P_< \circ P_{\pm}^{\mathrm{cyl}} \circ \diamond^{\mathrm{cyl}}|_{\Omega_{\mathbb{K}}^{<}(\hat{M}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \Omega_{\mathbb{K}}^{<}(\hat{M})} ,$$
$$\blacklozenge_{\pm}^{\mathrm{cone}} \stackrel{\mathrm{def.}}{=} 2P_< \circ P_{\pm}^{\mathrm{cone}} \circ \diamond^{\mathrm{cone}}|_{\Omega_{\mathbb{K}}^{<}(\hat{M}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \Omega_{\mathbb{K}}^{<}(\hat{M})} .$$

Combining these relations with (2.12), (2.23) and with the commutation relations given above, one easily checks the identity:

$$r^{\mathcal{E}} \circ \blacklozenge_{\pm}^{\mathrm{cyl}} = \blacklozenge_{\pm}^{\mathrm{cone}} \circ (r^{\mathcal{E}} \otimes r^{\mathcal{E}}) ,$$

which shows that we have mutually-inverse $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-linear unital isomorphisms of algebras:

$$(\Omega_{\mathbb{K}}^{<}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cyl}}) \underset{r^{-\mathcal{E}}|_{\Omega_{\mathbb{K}}^{<}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega_{\mathbb{K}}^{<}(\hat{M})}}{\rightleftarrows}} (\Omega_{\mathbb{K}}^{<}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cone}}) . \tag{2.32}$$

Relations (2.25) and (2.31) together with (2.16) and (2.17) show that $\mathcal{L}_{\partial_u}$ and $\mathcal{L}_{\partial_u} - \mathcal{E}$ are derivations of the the algebras $(\Omega_{\mathbb{K}}^{<}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cyl}})$ and $(\Omega_{\mathbb{K}}^{<}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cone}})$, respectively. The observations made above show that the subspaces of *truncated special inhomogeneous forms*:

$$\Omega_{\mathbb{K}}^{<,\mathrm{cyl}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \Omega_{\mathbb{K}}^{<}(\hat{M}) \cap \Omega_{\mathbb{K}}^{\mathrm{cyl}}(\hat{M}), \quad \Omega_{\mathbb{K}}^{<,\mathrm{cone}}(\hat{M}) \stackrel{\mathrm{def.}}{=} \Omega_{\mathbb{K}}^{<}(\hat{M}) \cap \Omega_{\mathbb{K}}^{\mathrm{cone}}(\hat{M})$$

are unital $\mathcal{C}_{\perp}^\infty(\hat{M}, \mathbb{K})$-subalgebras of the algebras $(\Omega_{\mathbb{K}}^{<}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cyl}})$ and $(\Omega_{\mathbb{K}}^{<}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cone}})$. Furthermore:

Proposition. We have mutually-inverse unital isomorphisms of $\mathbb{K}$-algebras:

$$(\Omega_{\mathbb{K}}^{<,\mathrm{cyl}}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cyl}}) \underset{r^{-\mathcal{E}}|_{\Omega_{\mathbb{K}}^{<,\mathrm{cone}}(\hat{M})}}{\overset{r^{\mathcal{E}}|_{\Omega_{\mathbb{K}}^{<,\mathrm{cyl}}(\hat{M})}}{\rightleftarrows}} (\Omega_{\mathbb{K}}^{<,\mathrm{cone}}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cone}}) \quad . \tag{2.33}$$

The situation for the cylinder is summarized in the following commutative diagram:

$$\begin{tikzcd}
(\Omega_{\mathbb{K}}^{<,\mathrm{cyl}}(\hat{M}), \blacklozenge_{\pm}^{\mathrm{cyl}}) \arrow[r, shift left, "P_{\pm}^{\mathrm{cyl}}"] \arrow[d, shift left, "\Xi_{\pm}^{\mathrm{cyl}}"] & (\Omega_{\mathbb{K}}^{\pm,\mathrm{cyl}}(\hat{M}), \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "2P_<"] \arrow[d, shift left, "2P_\perp"] \\
(\Omega_{\mathbb{K}}(M), \diamond) \arrow[u, shift left, "(\Xi_{\pm}^{\mathrm{cyl}})^{-1}"] \arrow[r, shift left, "\Pi^*"] & (\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M}), \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "j^*"] \arrow[u, shift left, "P_{\pm}^{\mathrm{cyl}}"]
\end{tikzcd} \tag{2.34}$$

where $\Xi^{\rm cyl}_\pm \stackrel{\rm def.}{=} 2j^* \circ P_\perp \circ P^{\rm cyl}_\pm$ and $(\Xi^{\rm cyl}_\pm)^{-1} = 2P_< \circ P^{\rm cyl}_\pm \circ \Pi^*$, while the situation for the cone is summarized in the commutative diagram:

$$\begin{tikzcd}
(\Omega^{<,\rm cone}_{\mathbb{K}}(\hat{M}), \blacklozenge^{\rm cone}_\pm) \arrow[r, shift left, "P^{\rm cone}_\pm"] \arrow[d, shift left, "\Xi^{\rm cone}_\pm"] & (\Omega^{\pm,\rm cone}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone}) \arrow[l, shift left, "2P_<"] \arrow[d, shift left, "2P_\perp"] \\
(\Omega_{\mathbb{K}}(M), \diamond) \arrow[u, shift left, "(\Xi^{\rm cone}_\pm)^{-1}"] \arrow[r, shift left, "r^{\mathcal{E}}\circ\, \Pi^*"] & (\Omega^{\perp,\rm cone}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone}) \arrow[l, shift left, "j^*\circ\, r^{-\mathcal{E}}"] \arrow[u, shift left, "P^{\rm cone}_\pm"]
\end{tikzcd} \tag{2.35}$$

where $\Xi^{\rm cone}_\pm \stackrel{\rm def.}{=} 2j^* \circ\ r^{-\mathcal{E}} \circ P_\perp \circ P^{\rm cone}_\pm$ and $(\Xi^{\rm cone}_\pm)^{-1} = 2P_< \circ P^{\rm cone}_\pm \circ r^{\mathcal{E}} \circ \Pi^*$. The full collection of isomorphic models of the Kähler-Atiyah algebra of (M, g) which arise from the cone and cylinder constructions is summarized in the commutative diagram below:

$$\begin{tikzcd}
 & (\Omega^{<,\rm cyl}_{\mathbb{K}}(\hat{M}), \blacklozenge^{\rm cyl}_\pm) \arrow[rr, shift left, "P^{\rm cyl}_\pm"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \arrow[dd, dotted, shift left, "\Xi^{\rm cyl}_\pm"] & & (\Omega^{\pm,\rm cyl}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl}) \arrow[ll, shift left, "2P_<"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \arrow[dd, shift left, "2P_\perp"] \\
(\Omega^{<,\rm cone}_{\mathbb{K}}(\hat{M}), \blacklozenge^{\rm cone}_\pm) \arrow[ur, shift left, "r^{-\mathcal{E}}"] \arrow[rr, shift left, "P^{\rm cone}_\pm"] \arrow[dd, shift left, "\Xi^{\rm cone}_\pm"] & & (\Omega^{\pm,\rm cone}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone}) \arrow[ll, shift left, "2P_<"] \arrow[ur, shift left, "r^{-\mathcal{E}}"] \arrow[dd, shift left, "2P_\perp"] & \\
 & (\Omega_{\mathbb{K}}(M), \diamond) \arrow[uu, dotted, shift left, "(\Xi^{\rm cyl}_\pm)^{-1}"] \arrow[rr, dotted, shift left, "\Pi^*"] \arrow[dl, dotted, shift left, "\rm id"] & & (\Omega^{\perp,\rm cyl}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl}) \arrow[ll, dotted, shift left, "j^*"] \arrow[uu, shift left, "P^{\rm cyl}_\pm"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \\
(\Omega_{\mathbb{K}}(M), \diamond) \arrow[uu, shift left, "(\Xi^{\rm cone}_\pm)^{-1}"] \arrow[ur, dotted, shift left, "\rm id"] \arrow[rr, shift left, "r^{\mathcal{E}}\circ\, \Pi^*"] & & (\Omega^{\perp,\rm cone}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cone}) \arrow[ll, shift left, "j^*\circ\, r^{-\mathcal{E}}"] \arrow[uu, shift left, "P^{\rm cone}_\pm"] \arrow[ur, shift left, "r^{-\mathcal{E}}"] &
\end{tikzcd}$$

2.3 The Levi-Civita connections of the cylinder and cone

The Levi-Civita connections. Using the formulas for the Levi-Civita connection of a warped product, one finds that the Levi-Civita connection of the cylinder is given by:

$$\nabla^{\rm cyl}_V W = V(g_{\rm cyl}(\partial_u, W))\partial_u + \nabla^*_V(W^\perp)\,, \quad \forall V, W \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})\,,$$

where $W^\perp \in \Gamma(\hat{M}, T^\perp_{\mathbb{K}}\hat{M})$ is the part of W orthogonal to ∂_u (and thus to ∂_r):

$$W^\perp = W - g_{\rm cyl}(\partial_u, W)\partial_u = W - g_{\rm cone}(\partial_r, W)\partial_r$$

while ∇^* (a connection on the bundle $T^\perp_{\mathbb{K}}\hat{M} \approx \Pi^*(T_{\mathbb{K}}M)$) is the pullback along Π of the Levi-Civita connection ∇ of (M, g). Notice the properties:

$$\nabla^{\rm cyl}_V(W)^\perp = \nabla^{\rm cyl}_V(W^\perp) = \nabla^*_V(W^\perp)\,, \quad \nabla^{\rm cyl}_V(\partial_u) = 0\,, \quad \forall V, W \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})\ . \tag{2.36}$$

On the other hand, the Levi-Civita connection of the cone can be expressed as:

$$\nabla^{\rm cone}_V W = \frac{1}{r}\nabla^{\rm cyl}_V(rW) + \lambda(V, W)\,, \quad \forall\ V, W \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})\,, \tag{2.37}$$

where $\lambda \in \Gamma(\hat{M}, T^*_{\mathbb{K}}\hat{M} \otimes T^*_{\mathbb{K}}\hat{M} \otimes T_{\mathbb{K}}\hat{M})$ is a tensor of type $(2,1)$ (i.e. with two covariant indices and one contravariant index) defined through the formula:

$$\lambda(V,W) = g_{\mathrm{cyl}}(\partial_u, W)V - g_{\mathrm{cyl}}(V,W)\partial_u = \frac{1}{r}(g_{\mathrm{cone}}(\partial_r, W)V - g_{\mathrm{cone}}(V,W)\partial_r)\,, \qquad (2.38)$$

for all $V, W \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$. In particular, the Levi-Civita connections of the cylinder and cone satisfy the following relations for all vector fields X, Y on M:

$$\begin{aligned} &\nabla^{\mathrm{cyl}}_{X_*} Y_* = (\nabla_X Y)_*\,, && \nabla^{\mathrm{cyl}}_{\partial_u}\partial_u = 0\,, \quad \nabla^{\mathrm{cyl}}_{\partial_u} X_* = \nabla^{\mathrm{cyl}}_{X_*}\partial_u = 0\,, \quad (2.39)\\ &\nabla^{\mathrm{cone}}_{X_*} Y_* = (\nabla_X Y)_* - (g(X,Y)\circ \Pi) r\partial_r\,, && \nabla^{\mathrm{cone}}_{\partial_r}\partial_r = 0\,, \quad \nabla^{\mathrm{cone}}_{\partial_r} X_* = \nabla_{X_*}\partial_r = \frac{1}{r} X_*\,. \end{aligned}$$

Remark. We note that λ can also be written in the form:

$$\lambda(V,W) = g_{\mathrm{cyl}}(\partial_u, W)V^\perp - g_{\mathrm{cyl}}(V^\perp, W)\partial_u = \frac{1}{r}(g_{\mathrm{cone}}(\partial_r, W)V^\perp - g_{\mathrm{cone}}(V^\perp, W)\partial_r)\,,$$

where $V, W \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$ and $V^\perp$ is the part of V which is orthogonal to ∂_r (and hence to ∂_u).

The connections induced on differential forms. Direct computation using (2.37) shows that the connections induced on $\Omega_{\mathbb{K}}(\hat{M})$ by the Levi-Civita connections of the cone and cylinder are related through:

$$\nabla^{\mathrm{cone}}_V \omega = (r^{\mathcal{E}} \circ \nabla^{\mathrm{cyl}}_V \circ r^{-\mathcal{E}})(\omega) + \frac{1}{r}\left[V_{\#_{\mathrm{cone}}} \wedge (\partial_r \lrcorner \omega) - \theta \wedge (V \lrcorner \omega)\right]\,, \quad \forall V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})\,,$$
$$\forall \omega \in \Omega_{\mathbb{K}}(\hat{M}) \qquad (2.40)$$

where $V_{\#_{\mathrm{cone}}}$ is the one-form dual to V with respect to the cone metric:

$$V_{\#_{\mathrm{cone}}} = V \lrcorner g_{\mathrm{cone}}\,.$$

Remark. One has the obvious identity:

$$V^{\|}_{\#_{\mathrm{cone}}} \wedge (\partial_r \lrcorner \omega) = \theta \wedge (V^{\|} \lrcorner \omega)\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(\hat{M})\,,$$

where $V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$ and $V^{\|} = g_{\mathrm{cyl}}(\partial_u, V)\partial_u = g_{\mathrm{cone}}(\partial_r, V)\partial_r$ is the part of V which is parallel to ∂_u (and thus to ∂_r). This implies that (2.40) can also be written as:

$$\nabla^{\mathrm{cone}}_V \omega = (r^{\mathcal{E}} \circ \nabla^{\mathrm{cyl}}_V \circ r^{-\mathcal{E}})(\omega) + \frac{1}{r}\left[V^\perp_{\#_{\mathrm{cone}}} \wedge (\partial_r \lrcorner \omega) - \theta \wedge (V^\perp \lrcorner \omega)\right]\,, \quad \forall V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})\,,$$
$$\forall \omega \in \Omega_{\mathbb{K}}(\hat{M}) \qquad (2.41)$$

where $V^\perp_{\#_{\mathrm{cone}}} = V^\perp \lrcorner g_{\mathrm{cone}}$. Using the expansion of the graded $\diamond$-commutator (see section 3 of [7]), it is not very hard to check that the following identities hold for all $\omega \in \Omega_{\mathbb{K}}(\hat{M})$ and any vector field $V^\perp \in \Gamma(\hat{M}, T^\perp_{\mathbb{K}}\hat{M})$ which is everywhere orthogonal to ∂_r (and thus to ∂_u as well):

$$\begin{aligned} V^\perp_{\#_{\mathrm{cone}}} \wedge (\partial_r \lrcorner \omega) - \theta \wedge (V^\perp \lrcorner \omega) = -(V^\perp_{\#_{\mathrm{cone}}} \wedge \theta) \bigtriangleup^{\mathrm{cone}}_1 \omega &= \frac{1}{2}[[V^\perp_{\#_{\mathrm{cone}}} \wedge \theta, \omega]]_{-,\diamond^{\mathrm{cone}}} \\ &= \frac{1}{2}[V^\perp_{\#_{\mathrm{cone}}} \wedge \theta, \omega]_{-,\diamond^{\mathrm{cone}}} \end{aligned}$$

and:

$$V^{\perp}_{\#_{\rm cone}} \wedge (\partial_r \lrcorner \omega) - \theta \wedge (V^{\perp} \lrcorner \omega) = -r(V^{\perp}_{\#_{\rm cyl}} \wedge \psi) \bigtriangleup_1^{\rm cyl} \omega = \frac{r}{2}[[V^{\perp}_{\#_{\rm cyl}} \wedge \psi, \omega]]_{-,\diamond^{\rm cyl}}$$
$$= \frac{r}{2}[V_{\#_{\rm cyl}} \wedge \psi, \omega]_{-,\diamond^{\rm cyl}} .$$

Combining the last identity with (2.41) gives the following relation which will be used below:

$$\begin{aligned} \nabla^{\rm cone}_V \omega &= (r^{\mathcal{E}} \circ \nabla^{\rm cyl}_V \circ r^{-\mathcal{E}})(\omega) + \frac{1}{2}[V^{\perp}_{\#_{\rm cyl}} \wedge \psi, \omega]_{-,\diamond^{\rm cyl}} \\ &= (r^{\mathcal{E}} \circ \nabla^{\rm cyl}_V \circ r^{-\mathcal{E}})(\omega) + \frac{1}{2r}[V^{\perp}_{\#_{\rm cone}} \wedge \theta, \omega]_{-,\diamond^{\rm cone}} \end{aligned} \tag{2.42}$$

Equation (2.42) expresses $\nabla^{\rm cone}$ in terms of $\nabla^{\rm cyl}$ using operations from the Kähler-Atiyah algebra of the cylinder or of the cone.

Some useful identities. The identity:

$$\nabla^{\rm cyl}_V (W \lrcorner \omega) = (\nabla^{\rm cyl}_V W) \lrcorner \omega + W \lrcorner \nabla^{\rm cyl}_V \omega$$

(which is also satisfied by any linear connection on $T_{\mathbb{K}}\hat{M}$) and the second relation in (2.36) imply:

$$[\nabla^{\rm cyl}_V, \partial_u \lrcorner\ ]_{-,\circ} = 0 \Longleftrightarrow [\nabla^{\rm cyl}_V, \iota^{\rm cyl}_{\psi}]_{-,\circ} = 0 \,, \quad \forall V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M}) \,,$$

while the fact that $\nabla^{\rm cyl}_V$ is an even derivation of the exterior algebra of $\hat{M}$ and the obvious relation $\nabla^{\rm cyl}_V \psi = 0$ imply:

$$[\nabla^{\rm cyl}_V, \wedge_{\psi}]_{-,\circ} = 0 \,, \quad \forall V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M}) \ .$$

These two properties of $\nabla^{\rm cyl}$ imply that the following identities hold for all $V, W \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$:

$$[\nabla^{\rm cyl}_V, P_{||}]_{-,\circ} = [\nabla^{\rm cyl}_V, P_{\perp}]_{-,\circ} = 0 \Longrightarrow \nabla^{\rm cyl}_V(\omega_{||}) = \nabla^{\rm cyl}_V(\omega)_{||} \,, \ \nabla^{\rm cyl}_V(\omega_{\perp}) = \nabla^{\rm cyl}_V(\omega)_{\perp} \,, \ \forall \omega \in \Omega_{\mathbb{K}}(\hat{M})$$

as well as:

$$\nabla^{\rm cyl}_V(\omega^{\rm cyl}_{\top}) = \nabla^{\rm cyl}_V(\omega)^{\rm cyl}_{\top} \,, \quad \forall \omega \in \Omega_{\mathbb{K}}(\hat{M}) \ .$$

Using these observations and the fact that $\nabla^{\rm cyl}_V \nu^{\rm cyl} = 0$, we find that the morphisms $\varphi^{\rm cyl}_{\pm}$ constructed using $\diamond^{\rm cyl}$ and $\nu^{\rm cyl}$ as in section 3.10 of [7] satisfy the following relation which will be used later on:

$$[\nabla^{\rm cyl}_V, \varphi^{\rm cyl}_{\pm}]_{-,\circ} = 0 \Longleftrightarrow \nabla^*_V \circ \varphi^{\rm cyl}_{\pm} = \varphi^{\rm cyl}_{\pm} \circ \nabla^{\rm cyl}_V \,, \quad \forall V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M}) \,, \tag{2.43}$$

where we used the fact that the restriction $\nabla^{\rm cyl}|_{\wedge(T^{\perp}_{\mathbb{K}}\hat{M})^*}$ equals the pullback ∇^* through Π of the connection induced by ∇ on $\wedge T^*_{\mathbb{K}}M$ (as usual, we identify $\wedge(T^{\perp}_{\mathbb{K}}\hat{M})^* \approx \Pi^*(\wedge T^*_{\mathbb{K}}M)$.

2.4 Pinors on metric cylinders and cones

The pin bundle of $\hat{M}$. Let S be a pin bundle of (M,g) and $\gamma : (\wedge T^*_{\mathbb{K}}M, \diamond) \to (\mathrm{End}(M), \circ)$ be its fiberwise representation. Let $\hat{S} \stackrel{\text{def.}}{=} \Pi^*(S)$ be the pullback bundle and $\gamma_* \stackrel{\text{def.}}{=} \Pi^*(\gamma) : \wedge(T^\perp_{\mathbb{K}}\hat{M})^* \to \mathrm{End}(\hat{S})$ be the pullback of γ to the bundle $\Pi^*(\wedge T^*_{\mathbb{K}}M) \approx \wedge(T^\perp_{\mathbb{K}}\hat{M})^*$. Recall that our assumptions imply that γ induces a bijection from $\Omega_{\mathbb{K}}(M) = \Gamma(M, \wedge T^*_{\mathbb{K}}M)$ to $\Gamma(M, \mathrm{End}(S))$. In turn, this implies that the map induced by γ_* on sections is an isomorphism of $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-algebras:

$$\gamma_* : \Omega^\perp_{\mathbb{K}}(\hat{M}) \xrightarrow{\sim} \Gamma(\hat{M}, \mathrm{End}(\hat{S}))\ .$$

We have the basic property:

$$\gamma_* \circ \Pi^* = \Pi^* \circ \gamma\,, \tag{2.44}$$

where, in the left hand side, Π^* denotes the pullback of differential forms while in the right hand side it denotes the pullback of sections of $\mathrm{End}(S)$ to sections of $\mathrm{End}(\hat{S})$. This gives a commutative square of unital morphisms of $\mathcal{C}^\infty(M, \mathbb{K})$-algebras which constitutes the rightmost part of the diagram (2.45) (as usual, we identify $\mathcal{C}^\infty_\perp(\hat{M}, \mathbb{K}) \approx \mathcal{C}^\infty(M, \mathbb{K})$).

$$\begin{tikzcd} \Omega_{\mathbb{K}}(\hat{M}) \arrow[r, "\varphi^{\mathrm{cyl}}_\epsilon"] \arrow[d, "r^{\mathcal{E}}"'] \arrow[dr, "\gamma_{\mathrm{cyl}}"] & \Omega^\perp_{\mathbb{K}}(\hat{M}) \arrow[d, "\gamma_*"] & \Omega_{\mathbb{K}}(M) \arrow[l, "\Pi^*"'] \arrow[d, "\gamma"] \\ \Omega_{\mathbb{K}}(\hat{M}) \arrow[r, "\gamma_{\mathrm{cone}}"] & \Gamma(\hat{M}, \mathrm{End}(\hat{S})) & \Gamma(M, \mathrm{End}(S)) \arrow[l, "\Pi^*"'] \end{tikzcd} \tag{2.45}$$

The morphisms γ_{cyl} and γ_{cone}. Let us fix a sign factor $\epsilon \in \{-1, 1\}$. Considering the unital morphisms of $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-algebras:

$$\begin{aligned} \varphi^{\mathrm{cyl}}_\epsilon &= 2P_\perp \circ P^{\mathrm{cyl}}_\epsilon : (\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}}) \to (\Omega^\perp_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}})\,, \\ \varphi^{\mathrm{cone}}_\epsilon &= 2P_\perp \circ P^{\mathrm{cone}}_\epsilon : (\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}}) \to (\Omega^\perp_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}}) \end{aligned}$$

as in [7], we define unital morphisms of $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-algebras: $\gamma_{\substack{\mathrm{cyl}\\ \mathrm{cone}}} : (\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\substack{\mathrm{cyl}\\ \mathrm{cone}}}) \to (\Gamma(\hat{M}, \mathrm{End}(\hat{S})), \circ)$ through (see diagram (2.45)):

$$\gamma_{\mathrm{cyl}} \stackrel{\text{def.}}{=} \gamma_* \circ \varphi^{\mathrm{cyl}}_\epsilon\,, \quad \gamma_{\mathrm{cone}} \stackrel{\text{def.}}{=} \gamma_{\mathrm{cyl}} \circ r^{-\mathcal{E}}\ . \tag{2.46}$$

It is clear that $\gamma_{\substack{\mathrm{cyl}\\ \mathrm{cone}}}$ are $\mathcal{C}^\infty(\hat{M}, \mathbb{K})$-linear, so they are induced by corresponding morphisms of bundles of algebras, which are easily seen to be irreducible on the fibers. Since $r^{\mathcal{E}}$ commutes with $P_\perp$ and satisfies (2.23), we find:

$$r^{\mathcal{E}} \circ \varphi^{\mathrm{cyl}}_\epsilon = \varphi^{\mathrm{cone}}_\epsilon \circ r^{\mathcal{E}} \Longleftrightarrow \varphi^{\mathrm{cyl}}_\epsilon \circ r^{-\mathcal{E}} = r^{-\mathcal{E}} \circ \varphi^{\mathrm{cone}}_\epsilon\,,$$

which means that γ_{cone} can also be written as (see diagram (2.47)):

$$\gamma_{\mathrm{cone}} = \gamma_* \circ r^{-\mathcal{E}} \circ \varphi^{\mathrm{cone}}_\epsilon\ .$$

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$$
\begin{tikzcd}
\Omega_{\mathbb{K}}(\hat{M}) \arrow[rr, "r^{\mathcal{E}}"] \arrow[dr, "\gamma_{\mathrm{cyl}}"] \arrow[dd, "\varphi_\epsilon^{\mathrm{cyl}}"'] & & \Omega_{\mathbb{K}}(\hat{M}) \arrow[dl, "\gamma_{\mathrm{cone}}"'] \arrow[dd, "\varphi_\epsilon^{\mathrm{cone}}"] \\
& \Gamma(\hat{M}, \mathrm{End}(\hat{S})) & \\
\Omega^{\perp}_{\mathbb{K}}(\hat{M}) \arrow[ur, "\gamma_*"] \arrow[rr, "r^{\mathcal{E}}"] & & \Omega^{\perp}_{\mathbb{K}}(\hat{M}) \arrow[ul, "\gamma_* \circ\, r^{-\mathcal{E}}"']
\end{tikzcd}
\tag{2.47}
$$

The last relation of section 3.10 in [7] gives $\varphi_\epsilon^{\mathrm{cyl}}(\nu^{\mathrm{cyl}}) = \epsilon 1_{\hat{M}}$, which implies $\gamma_{\mathrm{cyl}}(\nu^{\mathrm{cyl}}) = \epsilon\gamma_*(1_{\hat{M}}) = \epsilon\Pi^*(\gamma(1_M)) = \epsilon\Pi^*(\mathrm{id}_S) = \epsilon\ \mathrm{id}_{\hat{S}}$, where we noticed that $1_{\hat{M}} = \Pi^*(1_M)$ and $\mathrm{id}_{\hat{S}} = \Pi^*(\mathrm{id}_S)$. We find:

$$\gamma_{\mathrm{cyl}}(\nu^{\mathrm{cyl}}) = \gamma_{\mathrm{cone}}(\nu^{\mathrm{cone}}) = \epsilon\ \mathrm{id}_{\hat{S}}\,, \tag{2.48}$$

where we used the fact that $r^{-\mathcal{E}}(\nu^{\mathrm{cone}}) = \nu^{\mathrm{cyl}}$. It follows that γ_{cyl} makes $\hat{S}$ into a pin bundle of $(\hat{M}, g_{\mathrm{cyl}})$ having signature ϵ, while γ_{cone} makes $\hat{S}$ into a pin bundle of $(\hat{M}, g_{\mathrm{cone}})$ of the same signature. Since $\varphi_\epsilon^{\mathrm{cyl}}(\psi) = \epsilon\nu_\top^{\mathrm{cyl}} = \epsilon\Pi^*(\nu)$, we also have $\gamma_{\mathrm{cone}}(\theta) = \gamma_{\mathrm{cyl}}(\psi) = \epsilon\gamma_*(\Pi^*(\nu)) = \epsilon\ \Pi^*(\gamma(\nu))$, where we noticed that $r^{-\mathcal{E}}(\theta) = \psi$. Thus:

$$\gamma_{\mathrm{cone}}(\theta) = \gamma_{\mathrm{cyl}}(\psi) = \epsilon\ \Pi^*(\gamma(\nu))\ . \tag{2.49}$$

The form of φ_ϵ given in [7] implies:

$$\gamma_{\mathrm{cyl}}(\omega) = \gamma_*(\epsilon\tilde{*}_0^{\mathrm{cyl}}(\omega_\top^{\mathrm{cyl}}) + \omega_\perp)\,, \quad \gamma_{\mathrm{cone}}(\omega) = (\gamma_* \circ r^{-\mathcal{E}})(\epsilon\tilde{*}_0^{\mathrm{cone}}(\omega_\top^{\mathrm{cone}}) + \omega_\perp)\,,$$

where:

$$\omega_\top^{\mathrm{cyl}} \overset{\mathrm{def.}}{=} \iota_\psi^{\mathrm{cyl}}\omega = \partial_u \lrcorner \omega\,, \quad \omega_\top^{\mathrm{cone}} \overset{\mathrm{def.}}{=} \iota_\theta^{\mathrm{cone}}\omega = \partial_r \lrcorner \omega = \frac{1}{r}\omega_\top^{\mathrm{cyl}}\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(\hat{M})\ .$$

Furthermore, we have the following formulas for the cylinder and cone lifts (2.29) and (2.30) of a form $\omega \in \Omega_{\mathbb{K}}(M)$:

$$\gamma_{\mathrm{cyl}}(\omega_{\mathrm{cyl}}) = \gamma_{\mathrm{cone}}(\omega_{\mathrm{cone}}) = \Pi^*(\gamma(\omega))\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(M)\,, \tag{2.50}$$

where we used the fact that $\Pi^*(\omega) \in \Omega_{\mathbb{K}}^{\perp}(\hat{M})$ while $\varphi_\epsilon^{\mathrm{cyl}}$ restricts to the identity on $\Omega_{\mathbb{K}}^{\perp}(\hat{M})$.

The dequantization maps of the cylinder and cone. Since $P_\perp$ restricts to a bijection from $\Omega^\epsilon_{\mathbb{K}, \substack{\mathrm{cyl} \\ \mathrm{cone}}}(\hat{M})$ to $\Omega_{\mathbb{K}}^{\perp}(\hat{M})$, the restriction of $\gamma_{\substack{\mathrm{cyl} \\ \mathrm{cone}}}$ to $\Omega^\epsilon_{\mathbb{K}, \substack{\mathrm{cyl} \\ \mathrm{cone}}}(\hat{M})$ is a composition of bijections:

$$\gamma_{\mathrm{cyl}}|_{\Omega^\epsilon_{\mathbb{K},\mathrm{cyl}}(\hat{M})} = 2\gamma_*|_{\Omega^\perp_{\mathbb{K}}(\hat{M})} \circ P_\perp|^{\Omega^\perp_{\mathbb{K}}(\hat{M})}_{\Omega^\epsilon_{\mathbb{K},\mathrm{cyl}}(\hat{M})}\,, \quad \gamma_{\mathrm{cone}}|_{\Omega^\epsilon_{\mathbb{K},\mathrm{cone}}(\hat{M})} = 2\gamma_*|_{\Omega^\perp_{\mathbb{K}}(\hat{M})} \circ r^{-\mathcal{E}} \circ P_\perp|^{\Omega^\perp_{\mathbb{K}}(\hat{M})}_{\Omega^\epsilon_{\mathbb{K},\mathrm{cone}}(\hat{M})}$$

and hence the partial inverses of γ_{cyl} and γ_{cone} are given by:

$$\gamma_{\mathrm{cyl}}^{-1} = P_\epsilon^{\mathrm{cyl}} \circ \gamma_*^{-1}\,, \quad \gamma_{\mathrm{cone}}^{-1} = P_\epsilon^{\mathrm{cone}} \circ r^{\mathcal{E}} \circ \gamma_*^{-1} = r^{\mathcal{E}} \circ \gamma_{\mathrm{cyl}}^{-1}\,, \tag{2.51}$$

where we used the fact (see subsection 3.10 of [7]) that the inverse of $2P_\perp|^{\Omega^\perp_{\mathbb{K}}(\hat{M})}_{\Omega^\epsilon_{\mathbb{K},\substack{\mathrm{cyl}\\ \mathrm{cone}}}(\hat{M})}$ is given by $P_\epsilon^{\substack{\mathrm{cyl}\\ \mathrm{cone}}}|^{\Omega^\epsilon_{\mathbb{K},\substack{\mathrm{cyl}\\ \mathrm{cone}}}(\hat{M})}_{\Omega^\perp_{\mathbb{K}}(\hat{M})}$. The situation is summarized in diagram (2.52).

$$\begin{tikzcd}
\Gamma(\hat{M},\mathrm{End}(\hat{S})) \arrow[r, "\gamma_{\mathrm{cone}}^{-1}"] \arrow[d, "\gamma_*^{-1}"'] \arrow[dr, "\gamma_{\mathrm{cyl}}^{-1}"] & \Omega^\epsilon_{\mathbb{K},\mathrm{cone}}(\hat{M}) \\
\Omega^\perp_{\mathbb{K}}(\hat{M}) \arrow[r, "P_\epsilon^{\mathrm{cyl}}"'] & \Omega^\epsilon_{\mathbb{K},\mathrm{cyl}}(\hat{M}) \arrow[u, "r^{\mathcal{E}}"']
\end{tikzcd} \tag{2.52}$$

2.5 The Fierz isomorphism of cylinders and cones

The morphisms $\hat{E}$ and $\check{E}_*$. For the pin bundle S over M, let us consider, as in [7], the natural isomorphism $q : S\otimes S^* \xrightarrow{\sim} \mathrm{End}(S)$ as well as the isomorphism $\rho : S \xrightarrow{\sim} S^*$ induced by a non-degenerate admissible pairing $\mathscr{B}$ on S. The fiberwise bilinear pairing $\mathscr{B}$ pulls-back to a non-degenerate bilinear pairing $\hat{\mathscr{B}} \stackrel{\mathrm{def.}}{=} \Pi^*(\mathscr{B})$ on $\hat{S}$, which is easily seen to be admissible for both γ_{cyl} and γ_{cone}. It induces a bundle isomorphism between $\hat{S}$ and its dual which coincides with the pullback $\hat{\rho} \stackrel{\mathrm{def.}}{=} \Pi^*(\rho) : \hat{S} \xrightarrow{\sim} \hat{S}^*$ of ρ. Furthermore, the pullback $\hat{q} \stackrel{\mathrm{def.}}{=} \Pi^*(q) : \hat{S}\otimes\hat{S}^* \xrightarrow{\sim} \mathrm{End}(\hat{S})$ of q coincides with the natural isomorphism between $\hat{S}\otimes\hat{S}^*$ and $\mathrm{End}(\hat{S})$. Combining these, we find that the pullback:

$$\hat{E} \stackrel{\mathrm{def.}}{=} \Pi^*(E) = \hat{q}\circ(\mathrm{id}_{\hat{S}}\otimes\hat{\rho}) : \hat{S}\otimes\hat{S} \xrightarrow{\sim} \mathrm{End}(\hat{S}) \tag{2.53}$$

of the isomorphism $E = q\otimes(\mathrm{id}_S\otimes\rho) : S\otimes S \xrightarrow{\sim} \mathrm{End}(S)$ coincides with the isomorphism built as in [7] from the admissible bilinear pairing $\hat{\mathscr{B}}$ on $\hat{S}$. This implies that the bipinor bundle of algebras $(\hat{S}\otimes\hat{S}, \hat{\bullet})$ built from $\hat{S}$ using the pairing $\hat{\mathscr{B}}$ is the pullback of the bipinor bundle of algebras $(S\otimes S, \bullet)$ built from S using the pairing $\mathscr{B}$. Defining:

$$\check{E}_* \stackrel{\mathrm{def.}}{=} \gamma_*^{-1}\circ\hat{E} : \hat{S}\otimes\hat{S} \xrightarrow{\sim} \wedge(T^\perp_{\mathbb{K}}\hat{M})^* \approx \Pi^*(T^*_{\mathbb{K}}M)\,, \tag{2.54}$$

we have $\check{E}_* = \Pi^*(\check{E})$ where $\check{E} : S\otimes S \to \wedge T^*_{\mathbb{K}}M$ is the Fierz isomorphism of (M,g). The situation is summarized in the commutative diagram (2.55), which also encodes the action of $\check{E}_*$ on pulled-back forms.

$$\begin{tikzcd}
 & \Gamma(M,S\otimes S) \arrow[rr, "\mathrm{id}_S\otimes\rho"] \arrow[dl, "\Pi^*"'] \arrow[dd, dotted, "\check{E}"'] \arrow[ddrr, dotted, "E"] & & \Gamma(M,S\otimes S^*) \arrow[dl, "\Pi^*"'] \arrow[dd, "q"] \\
\Gamma(\hat{M},\hat{S}\otimes\hat{S}) \arrow[rr, "\mathrm{id}_{\hat{S}}\otimes\hat{\rho}"] \arrow[dd, "\check{E}_*"'] \arrow[ddrr, "\hat{E}"'] & & \Gamma(\hat{M},\hat{S}\otimes\hat{S}^*) \arrow[dd, "\hat{q}"] & \\
 & \Omega_{\mathbb{K}}(M) \arrow[dl, dotted, "\Pi^*"'] & & \Gamma(M,\mathrm{End}(S)) \arrow[ll, dotted, "\gamma^{-1}"'] \arrow[dl, "\Pi^*"] \\
\Omega^\perp_{\mathbb{K}}(\hat{M}) & & \Gamma(\hat{M},\mathrm{End}(\hat{S})) \arrow[ll, "\gamma_*^{-1}"] &
\end{tikzcd} \tag{2.55}$$

The relation between $\check{E}_*$ and $\check{E}$ is summarized in the smaller commutative diagram (2.56), where the pullback morphisms are non-surjective.

$$\begin{tikzcd}
\Gamma(\hat{M},\hat{S}\otimes\hat{S}) \arrow[rr, "\check{E}_*"] \arrow[dr, "\hat{E}"] & & \Omega^{\perp}_{\mathbb{K}}(\hat{M}) \arrow[dl, "\gamma_*"'] \\
& \Gamma(\hat{M},\mathrm{End}(\hat{S})) & \\
\Gamma(M,S\otimes S) \arrow[uu, "\Pi^*"] \arrow[rr, dotted, "\check{E}"] \arrow[dr, "E"'] & & \Omega_{\mathbb{K}}(M) \arrow[uu, "\Pi^*"'] \arrow[dl, "\gamma"] \\
& \Gamma(M,\mathrm{End}(S)) \arrow[uu, "\Pi^*"] &
\end{tikzcd} \tag{2.56}$$

In particular, we note the relations:

$$\hat{E}\circ\Pi^* = \Pi^*\circ E \tag{2.57}$$

and:

$$\check{E}_*\circ\Pi^* = \Pi^*\circ\check{E}\,, \tag{2.58}$$

which will be used later on.

The Fierz isomorphisms $\check{E}^{\mathrm{cyl}}$ and $\check{E}^{\mathrm{cone}}$. By definition, postcomposing $\hat{E}:\hat{S}\otimes\hat{S}\to\mathrm{End}(\hat{S})$ with the partial inverses $\gamma^{-1}_{\substack{\mathrm{cyl}\\ \mathrm{cone}}}:\mathrm{End}(\hat{S})\to\Omega^{\epsilon}_{\mathbb{K}}(\hat{M})$ defines the Fierz isomorphisms of the cylinder and cone:

$$\check{E}^{\mathrm{cyl}} \stackrel{\mathrm{def.}}{=} \gamma^{-1}_{\mathrm{cyl}}\circ\hat{E}:\hat{S}\otimes\hat{S}\xrightarrow{\sim}\wedge^{\epsilon,\mathrm{cyl}}T^*_{\mathbb{K}}\hat{M}\,,\quad \check{E}^{\mathrm{cone}} \stackrel{\mathrm{def.}}{=} \gamma^{-1}_{\mathrm{cone}}\circ\hat{E}:\hat{S}\otimes\hat{S}\xrightarrow{\sim}\wedge^{\epsilon,\mathrm{cone}}T^*_{\mathbb{K}}\hat{M}\,. \tag{2.59}$$

Relations (2.51) imply:

$$\check{E}^{\mathrm{cyl}} = P^{\mathrm{cyl}}_{\epsilon}\circ\check{E}_*\,,\quad \check{E}^{\mathrm{cone}} = P^{\mathrm{cone}}_{\epsilon}\circ r^{\mathcal{E}}\circ\check{E}_* = r^{\mathcal{E}}\circ P^{\mathrm{cyl}}_{\epsilon}\circ\check{E}_*\,. \tag{2.60}$$

as well as:

$$\check{E}^{\mathrm{cone}} = r^{\mathcal{E}}\circ\check{E}^{\mathrm{cyl}}\,.$$

Recall that our assumptions imply that $\mathrm{Cl}_{\mathbb{K}}(p,q)$ is simple and that its Schur algebra equals the base field. Therefore, the bundle morphism γ is a bundle isomorphism and the map which it induces on sections is bijective. As a consequence, the $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-linear map $\Omega^{\perp}_{\mathbb{K}}(\hat{M})\approx\Gamma(\hat{M},\wedge(T^{\perp}_{\mathbb{K}}\hat{M})^*)\to\Gamma(\hat{M},\mathrm{End}(\hat{S}))$ induced on sections (which, as usual, we have again denoted by γ_*) is bijective.

The pullback of pinors. The pullback of sections induces an injective but non-surjective $\mathcal{C}^\infty(M,\mathbb{K})$-linear map:

$$\Gamma(M,S)\xrightarrow{\Pi^*}\Gamma(\hat{M},\hat{S})\,, \tag{2.61}$$

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where, as usual, we identify $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K}) \approx \mathcal{C}^\infty(M,\mathbb{K})$. To characterize the image of this map, consider the following $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-submodule of the $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-module $\Gamma(\hat{M},\hat{S})$, which we shall call the $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-*module of vertical sections of* $\hat{S}$:

$$\Gamma^{\mathrm{vert}}(\hat{M},\hat{S}) \stackrel{\mathrm{def.}}{=} \{\hat{\xi} \in \Gamma(\hat{M},\hat{S}) | \mathcal{L}^{\hat{S}}_{\partial_u}\xi = 0\} = \{\hat{\xi} \in \Gamma(\hat{M},\hat{S}) | \mathcal{L}^{\hat{S}}_{\partial_r}\xi = 0\} \ .$$

Here and below, the symbol $\mathcal{L}^{\hat{S}}_V$ denotes the spinorial Lie derivative (a.k.a. the Kosmann-Schwarzbach derivative) [23, 24] of sections of $\hat{S}$ along a vector field $V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$. It is then easy to see that the image of (2.61) coincides with the $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-module of vertical pinors on $\hat{M}$:

$$\Pi^*(\Gamma(M,S)) = \Gamma^{\mathrm{vert}}(\hat{M},\hat{S}) \ .$$

Using the identification $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K}) \approx \mathcal{C}^\infty(M,\mathbb{K})$, the pullback of sections corestricts to an isomorphism of $\mathcal{C}^\infty(M,\mathbb{K})$-modules from $\Gamma(M,S)$ to $\Gamma^{\mathrm{vert}}(\hat{M},\hat{S})$, whose image is the appropriate restriction of the map $j^* : \Gamma(\hat{M},\hat{S}) \to \Gamma(M,S)$ which restricts sections to the closed submanifold $\{r = 1 \Leftrightarrow u = 0\} \approx M$ of $\hat{M}$:

$$\Gamma(M,S) \ \underset{j^*|_{\Gamma^{\mathrm{vert}}(\hat{M},\hat{S})}}{\overset{\Pi^*|_{\Gamma^{\mathrm{vert}}(\hat{M},\hat{S})}}{\rightleftarrows}} \ \Gamma^{\mathrm{vert}}(\hat{M},\hat{S}) \ . \tag{2.62}$$

Similarly, the pullback of sections gives an injective but non-surjective morphism of $\mathcal{C}^\infty(M,\mathbb{K})$-algebras (where we identify $\Pi^*(\mathrm{End}(S)) \approx \mathrm{End}(\hat{S})$ since $\hat{S} = \Pi^*(S)$):

$$(\Gamma(M,\mathrm{End}(S)),\circ) \xrightarrow{\Pi^*} (\Gamma(\hat{M},\mathrm{End}(\hat{S})),\circ) \, ,$$

whose image coincides with the following $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebra of $(\Gamma(\hat{M},\mathrm{End}(\hat{S})),\circ)$:

$$\begin{aligned}\Gamma^{\mathrm{vert}}(\hat{M},\mathrm{End}(\hat{S})) &\stackrel{\mathrm{def.}}{=} \{\hat{T} \in \Gamma(\hat{M},\mathrm{End}(\hat{S})) | [\mathcal{L}^{\hat{S}}_{\partial_u}, \hat{T}]_{-,\circ} = 0\} \\ &= \{\hat{T} \in \Gamma(\hat{M},\mathrm{End}(\hat{S})) | [\mathcal{L}^{\hat{S}}_{\partial_r}, \hat{T}]_{-,\circ} = 0\} \ .\end{aligned}$$

Pullback properties. Let $\xi, \xi' \in \Gamma(M,S)$ and $\xi_* = \Pi^*(\xi)$, $\xi'_* = \Pi^*(\xi') \in \Gamma(\hat{M},\hat{S})$ be their pullbacks. Relation (2.58) implies:

$$\check{E}^{\mathrm{cyl}} \circ \Pi^* = P^{\mathrm{cyl}}_\epsilon \circ \Pi^* \circ \check{E} \, , \quad \check{E}^{\mathrm{cone}} \circ \Pi^* = P^{\mathrm{cone}}_\epsilon \circ r^{\mathcal{E}} \circ \Pi^* \circ \check{E} \ . \tag{2.63}$$

This gives:

$$\begin{aligned}(\check{E}_*)_{\xi_*,\xi'_*} &= \Pi^*(\check{E})(\Pi^*(\xi \otimes \xi')) = \Pi^*(\check{E}(\xi \otimes \xi')) \\ &= \Pi^*(\check{E}_{\xi,\xi'}) \in \Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \Longrightarrow r^{\mathcal{E}}((\check{E}_*)_{\xi_*,\xi'_*}) \in \Omega^{\perp,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \, ,\end{aligned}$$

where we noticed that $\xi_* \otimes \xi'_* = \Pi^*(\xi \otimes \xi')$. It follows that:

$$\check{E}^{\mathrm{cyl}}_{\xi_*,\xi'_*} = P^{\mathrm{cyl}}_\epsilon(\Pi^*(\check{E}_{\xi,\xi'})) \in \Omega^{\epsilon,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \, , \quad \check{E}^{\mathrm{cone}}_{\xi_*,\xi'_*} = (P^{\mathrm{cone}}_\epsilon \circ r^{\mathcal{E}})(\Pi^*(\check{E}_{\xi,\xi'})) \in \Omega^{\epsilon,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \ .$$

The situation is summarized in the diagram below.

$$
\begin{tikzcd}
\Omega_{\mathbb{K}}(M) \arrow[rr, "\mathrm{id}_{\Omega_{\mathbb{K}}(M)}"] \arrow[dd, "\Pi^*"'] & & \Omega_{\mathbb{K}}(M) \arrow[dd, "\Pi^*"] \\
& \Gamma(M, S\otimes S) \arrow[ul, "\check{E}"] \arrow[ur, "\check{E}"'] \arrow[dd, "\Pi^*"' near start] & \\
\Omega^{\perp}_{\mathbb{K}}(\hat{M}) \arrow[rr, dotted, "r^{\mathcal{E}}"] \arrow[dd, "P^{\mathrm{cyl}}_{\epsilon}"'] & & \Omega^{\perp}_{\mathbb{K}}(\hat{M}) \arrow[dd, "P^{\mathrm{cone}}_{\epsilon}"] \\
& \Gamma(\hat{M}, \hat{S}\otimes \hat{S}) \arrow[ul, "\check{E}_*"] \arrow[ur, "r^{\mathcal{E}}\circ \check{E}_*"'] \arrow[dl, "\check{E}_{\mathrm{cyl}}"'] \arrow[dr, "\check{E}_{\mathrm{cone}}"] & \\
\Omega^{\epsilon}_{\mathbb{K},\mathrm{cyl}}(\hat{M}) \arrow[rr, "r^{\mathcal{E}}"] & & \Omega^{\epsilon}_{\mathbb{K},\mathrm{cone}}(\hat{M})
\end{tikzcd}
\tag{2.64}
$$

Notice that $\check{E}^{\substack{\mathrm{cyl}\\ \mathrm{cone}}}_{\xi_*,\xi'_*}$ lie in the corresponding $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-subalgebras of *special* twisted selfdual/anti-selfdual forms. As explained in the previous subsections, a computationally useful model for the later is provided by the $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebras $(\Omega^{<}_{\mathbb{K}}(\hat{M}), \blacklozenge_\epsilon^{\substack{\mathrm{cyl}\\ \mathrm{cone}}})$, which can therefore be used to implement the formalism of [7] in a symbolic computation system.

2.6 The connections on the pin bundle induced by the Levi-Civita connections of the cylinder and cone

The connection $\nabla^{\hat{S},\mathrm{cyl}}$ induced by ∇^{cyl} on $\hat{S}$ coincides with the pullback through Π of the connection ∇^S induced by ∇ on S:

$$\nabla^{\hat{S},\mathrm{cyl}} = (\nabla^S)^*\,, \tag{2.65}$$

while the connection $\nabla^{\hat{S},\mathrm{cone}}$ induced by ∇^{cone} on $\hat{S}$ is given by:

$$\nabla^{\hat{S},\mathrm{cone}}_V = \nabla^{\hat{S},\mathrm{cyl}}_V + \frac{1}{2}\gamma_{\mathrm{cyl}}(V^{\perp}_{\#_{\mathrm{cyl}}}\wedge\psi) = \nabla^{\hat{S},\mathrm{cyl}}_V + \frac{1}{2r}\gamma_{\mathrm{cone}}(V^{\perp}_{\#_{\mathrm{cone}}}\wedge\theta)\,,\ \forall V\in\Gamma(\hat{M},T^*_{\mathbb{K}}M)\,, \tag{2.66}$$

where we used the relation $\gamma_{\mathrm{cone}} = \gamma_{\mathrm{cyl}}\circ r^{-\mathcal{E}}$.

Remark. The Clifford connection property of $\nabla^{\hat{S},\mathrm{cyl}}$:

$$[\nabla^{\hat{S},\mathrm{cyl}}_V, \gamma_{\mathrm{cyl}}(\omega)]_{-,\circ} = \gamma_{\mathrm{cyl}}(\nabla^{\mathrm{cyl}}_V\omega)\,,\quad \forall\omega\in\Omega_{\mathbb{K}}(\hat{M})\,,\quad \forall V\in\Gamma(\hat{M},T_{\mathbb{K}}\hat{M})$$

follows from the definition (2.46) of γ_{cyl} upon using relation (2.43) as well as the identity:

$$[\nabla^{\hat{S},\mathrm{cyl}}_V, \gamma_*(\eta)] = [(\nabla^S)^*_V, \gamma_*(\eta)] = \gamma_*(\nabla^*_V\eta)\,,\quad \forall\eta\in\Omega^{\perp}_{\mathbb{K}}(\hat{M})\,,\quad \forall V\in\Gamma(\hat{M},T_{\mathbb{K}}\hat{M})\,,$$

which is a direct consequence of the Clifford connection property of ∇^S:

$$[\nabla^S_X, \gamma(\varrho)]_{-,\circ} = \gamma(\nabla_X \varrho)\,, \quad \forall \varrho \in \Omega_{\mathbb{K}}(M)\,, \quad \forall X \in \Gamma(M, T_{\mathbb{K}}M)\ .$$

On the other hand, the Clifford connection property of $\nabla^{\hat{S},\mathrm{cone}}$ follows from that of $\nabla^{\hat{S},\mathrm{cyl}}$ upon using equation (2.42). Indeed, for any $\omega \in \Omega_{\mathbb{K}}(\hat{M})$ and any $V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})$, we compute:

$$[\nabla^{\hat{S},\mathrm{cone}}_V, \gamma_{\mathrm{cone}}(\omega)]_{-,\circ} = [\nabla^{\hat{S},\mathrm{cyl}}_V, \gamma_{\mathrm{cyl}}(r^{-\mathcal{E}}\omega)]_{-,\circ} + \frac{1}{2}\gamma_{\mathrm{cyl}}([V^\perp_{\#_{\mathrm{cyl}}} \wedge \psi, r^{-\mathcal{E}}\omega]_{-,\diamond^{\mathrm{cyl}}})\,, \qquad (2.67)$$

where we used (2.66) and the fact that γ_{cyl} is a morphism of algebras. The first term in (2.67) equals $\gamma_{\mathrm{cyl}}(\nabla^{\mathrm{cyl}}(r^{-\mathcal{E}}\omega)) = \gamma_{\mathrm{cone}}((r^{\mathcal{E}} \circ \nabla^{\mathrm{cyl}} \circ r^{-\mathcal{E}})(\omega))$ by the Clifford connection property of $\nabla^{\hat{S},\mathrm{cyl}}$. Using the fact that $r^{\mathcal{E}}$ is a morphism of algebras from $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}})$ to $(\Omega_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cone}})$ and the relation $\gamma_{\mathrm{cone}} = \gamma_{\mathrm{cyl}} \circ r^{-\mathcal{E}}$, the second term of (2.67) can be expressed as:

$$\frac{r^2}{2}\gamma_{\mathrm{cone}}([V^\perp_{\#_{\mathrm{cyl}}} \wedge \psi, \omega]_{-,\diamond^{\mathrm{cone}}}) = \frac{1}{2r}\gamma_{\mathrm{cone}}([V^\perp_{\#_{\mathrm{cone}}} \wedge \theta, \omega]_{-,\diamond^{\mathrm{cone}}})\ .$$

Using these observations as well as identity (2.42), we see that the two terms in the right hand side of (2.67) combine to give:

$$[\nabla^{\hat{S},\mathrm{cone}}_V, \gamma_{\mathrm{cone}}(\omega)]_{-,\circ} = \gamma_{\mathrm{cone}}(\nabla^{\mathrm{cone}}_V \omega)\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(\hat{M})\,, \quad \forall V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})\,,$$

which is the Clifford property of $\nabla^{\hat{S},\mathrm{cone}}$. Hence the Clifford property of the canonical pin connection of the cone is a consequence of the rather subtle expression (2.42) for the connection induced on differential forms by the Levi-Civita connection of the cone.

Local expressions. Let $(e_m)_{m=1\ldots d}$ be an oriented local pseudo-orthonormal frame of (M,g). Then a convenient choice of oriented local pseudo-orthonormal frames and hence of their dual coframes for the cylinder and cone is given by:

$$\begin{aligned} e^{\mathrm{cyl}}_m &\stackrel{\mathrm{def.}}{=} (e_m)_*\,, & e^{\mathrm{cyl}}_{d+1} &\stackrel{\mathrm{def.}}{=} \partial_u = r\partial_r & &\Longleftrightarrow & e^m_{\mathrm{cyl}} &= \Pi^*(e^m)\,, & e^{d+1}_{\mathrm{cyl}} &= \psi\,, \\ e^{\mathrm{cone}}_m &\stackrel{\mathrm{def.}}{=} \frac{1}{r}(e_m)_*\,, & e^{\mathrm{cone}}_{d+1} &\stackrel{\mathrm{def.}}{=} \partial_r & &\Longleftrightarrow & e^m_{\mathrm{cone}} &= r\Pi^*(e^m)\,, & e^{d+1}_{\mathrm{cone}} &= \theta = r\psi\,, \end{aligned} \qquad (2.68)$$

where (e^m) is the coframe on M dual to (e_m) (thus $e^m(e_n) = \delta^m_n$). We have:

$$e^{\mathrm{cone}}_a = \frac{1}{r}e^{\mathrm{cyl}}_a \Longleftrightarrow e^a_{\mathrm{cone}} = re^a_{\mathrm{cyl}}\,, \quad \forall a = 1, \ldots, d+1\ .$$

Notice that $e^m_{\mathrm{cyl}}, e^m_{\mathrm{cone}}$ are the cylinder and cone lifts of the one-forms $e^m \in \Omega^1_{\mathbb{K}}(M)$ (see equations (2.29) and (2.30)). Let us define:

$$\gamma^m \stackrel{\mathrm{def.}}{=} \gamma(e^m) \in \Gamma(M, \mathrm{End}(S))\,, \quad \gamma^{(d+1)} \stackrel{\mathrm{def.}}{=} \gamma(\nu) \in \Gamma(M, \mathrm{End}(S))\ .$$

Since the local pseudo-orthonormal frame (e_m) of (M,g) is oriented, we have:

$$\gamma^{(d+1)} = \gamma^1 \circ \ldots \circ \gamma^d\ .$$

We also define:

$$\hat{\gamma}^a \stackrel{\text{def.}}{=} \gamma_{\text{cone}}(e^a_{\text{cone}}) = \gamma_{\text{cyl}}(e^a_{\text{cyl}}) \in \Gamma(\hat{M}, \text{End}(\hat{S}))\,,$$

where we used relation (2.46) and the fact that $r^{-\mathcal{E}}(e^a_{\text{cone}}) = e^a_{\text{cyl}}$. For simplicity of notation, we denote $\Pi^*(\gamma^m) = \gamma_*(e^{\text{cyl}}_m)$ by γ^m_* and $\Pi^*(\gamma^{(d+1)}) = \gamma_*(\nu^{\text{cyl}}_\top)$ by $\gamma^{(d+1)}_*$. Identities (2.50) and (2.49) give:

$$\hat{\gamma}^m = \Pi^*(\gamma^m) = \gamma^m_*\,, \quad \hat{\gamma}^{d+1} = \epsilon\, \Pi^*(\gamma^{(d+1)}) = \epsilon\, \gamma^{(d+1)}_*\,,$$

where we used the fact that $\varphi^{\text{cyl}}_\epsilon(e^m_{\text{cyl}}) = e^m_{\text{cyl}}$ since $\psi \perp e^m_{\text{cyl}}$. Relation (2.66) gives:

$$\nabla^{\hat{S},\text{cone}} = \nabla^{\hat{S},\text{cyl}} + \frac{1}{2} e^m_{\text{cyl}} \otimes \gamma_{\text{cyl}}((e^{\text{cyl}}_m)_{\#_{\text{cyl}}} \wedge \psi) = \nabla^{\hat{S},\text{cyl}} + \frac{1}{2r} e^m_{\text{cone}} \otimes \gamma_{\text{cone}}((e^{\text{cone}}_m)_{\#_{\text{cone}}} \wedge \theta)\,. \quad (2.69)$$

Combining with (2.65), we find:

$$\nabla^{\hat{S},\text{cone}}_{\partial_u} = \nabla^{\hat{S},\text{cyl}}_{\partial_u} = \mathcal{L}^{\hat{S}}_{\partial_u}\,, \quad \nabla^{\hat{S},\text{cone}}_{e^{\text{cyl}}_m} = \nabla^{\hat{S},\text{cyl}}_{e^{\text{cyl}}_m} + \frac{1}{2}\epsilon \gamma_{*,m} \gamma^{(d+1)}_*\,, \quad (2.70)$$

where $\mathcal{L}^{\hat{S}}_{\partial_u}$ is the Kosmann-Schwarzbach derivative with respect to ∂_u on $\hat{S}$ and:

$$\gamma_{*,m} \stackrel{\text{def.}}{=} \eta_{mn} \gamma^n_*\,.$$

Direct computation shows that the connection one-forms $\Omega_{mn} \stackrel{\text{def.}}{=} g(e_m, \nabla e_n)$ of ∇ in the frame (e_m) of M are related as follows to the connection one-forms $\Omega^{\text{cyl}}_{ab} \stackrel{\text{def.}}{=} g_{\text{cyl}}(e^{\text{cyl}}_a, \nabla^{\text{cyl}} e^{\text{cyl}}_b)$ of ∇^{cyl} and $\Omega^{\text{cone}}_{ab} \stackrel{\text{def.}}{=} g_{\text{cone}}(e^{\text{cone}}_a, \nabla^{\text{cone}} e^{\text{cone}}_b)$ of ∇^{cone} in the frames (e^{cyl}_a) and (e^{cone}_a) of $\hat{M}$, respectively:

$$\begin{aligned} &\Omega^{\text{cyl}}_{mn}(e^{\text{cyl}}_{d+1}) = 0\,, \quad \Omega^{\text{cyl}}_{mn}(e_p) = \Omega_{mn}(e_p)\,, \quad \Omega^{\text{cyl}}_{d+1m}(e_n) = 0\,, \qquad \Omega^{\text{cyl}}_{d+1m}(e^{\text{cyl}}_{d+1}) = 0 \\ &\Omega^{\text{cone}}_{mn}(e^{\text{cone}}_{d+1}) = 0\,, \quad \Omega^{\text{cone}}_{mn}(e_p) = \Omega_{mn}(e_p)\,, \quad \Omega^{\text{cone}}_{d+1m}(e_n) = -\eta_{mn}\,, \quad \Omega^{\text{cone}}_{d+1m}(e^{\text{cone}}_{d+1}) = 0\,. \end{aligned} \quad (2.71)$$

Using (2.71), it is easy to check that equations (2.70) agree with the formula $\nabla^{\hat{S},\,{}^{\text{cyl}}_{\text{cone}}} = \mathrm{d}^{\hat{S}} + \frac{1}{4}\Omega^{{}^{\text{cyl}}_{\text{cone}}}_{ab}\hat{\gamma}^{ab}$, where $\mathrm{d}^{\hat{S}} \stackrel{\text{def.}}{=} e^a_{\text{cyl}} \otimes \mathcal{L}^{\hat{S}}_{e^{\text{cyl}}_a} : \Gamma(\hat{M}, \hat{S}) \to \Omega^1_{\mathbb{K}}(\hat{M}, \hat{S})$ is the Kosmann-Schwarzbach differential of $\hat{S}$.

2.7 The lift of a general pin connection. Cone and cylinder dequantizations of the lift

Consider an arbitrary connection $D = \nabla^S + A$ on S, where $A \in \Omega^1_{\mathbb{K}}(M, \text{End}(S))$. We define the *lift* $\hat{D}$ of D to be the connection on $\hat{S}$ obtained from D by pullback through Π:

$$\hat{D} \stackrel{\text{def.}}{=} D^*\,. \quad (2.72)$$

Then $\hat{D}$ can be expressed as:

$$\hat{D} = \nabla^{\hat{S},\text{cyl}} + A^{\text{cyl}}\,,$$

where:

$$A^{\text{cyl}} \stackrel{\text{def.}}{=} \Pi^*(A) \in \Omega^1_{\mathbb{K}}(\hat{M}, \text{End}(\hat{S}))$$

and we used the fact that $(\nabla^S)^* = \nabla^{\hat{S},\mathrm{cyl}}$. Relation (2.69) implies that $\hat{D}$ can also be written in the form:

$$\hat{D} = \nabla^{\hat{S},\mathrm{cone}} + A^{\mathrm{cone}}\,, \tag{2.73}$$

where:

$$\begin{aligned} A^{\mathrm{cone}} &\stackrel{\mathrm{def.}}{=} A^{\mathrm{cyl}} - \frac{1}{2} e^m_{\mathrm{cyl}} \otimes \gamma_{\mathrm{cyl}}((e_m^{\mathrm{cyl}})_{\#_{\mathrm{cyl}}} \wedge \psi) \\ &= A^{\mathrm{cyl}} - \frac{1}{2r} e^m_{\mathrm{cone}} \otimes \gamma_{\mathrm{cone}}((e_m^{\mathrm{cone}})_{\#_{\mathrm{cone}}} \wedge \theta) \in \Omega^1_{\mathbb{K}}(\hat{M}, \mathrm{End}(\hat{S}))\,. \end{aligned}$$

The last relation amounts to:

$$\begin{aligned} A^{\mathrm{cone}}(V) &= A^{\mathrm{cyl}}(V) - \frac{1}{2}\gamma_{\mathrm{cyl}}((V^\perp)_{\#_{\mathrm{cyl}}} \wedge \psi) \\ &= A^{\mathrm{cyl}}(V) - \frac{1}{2r}\gamma_{\mathrm{cone}}((V^\perp)_{\#_{\mathrm{cone}}} \wedge \theta)\,, \quad \forall V \in \Gamma(\hat{M}, T_{\mathbb{K}}\hat{M})\,. \end{aligned}$$

Let us define:

$$\nabla^S_m \stackrel{\mathrm{def.}}{=} \nabla^S_{e_m}\,, \quad D_m \stackrel{\mathrm{def.}}{=} D_{e_m}$$

(which are derivations of the $\mathcal{C}^\infty(M,\mathbb{K})$-module $\Gamma(M,S)$) and:

$$A_m \stackrel{\mathrm{def.}}{=} A(e_m) \in \Gamma(M, \mathrm{End}(S))\,.$$

Then:

$$\nabla^S = \sum_{m=1}^d e^m \otimes \nabla^S_m\,, \quad D = \sum_{m=1}^d e^m \otimes D_m\,, \quad A = \sum_{m=1}^d e^m \otimes A_m \quad \text{and} \quad D_m = \nabla^S_m + A_m\,.$$

Similarly, we define:

$$\hat{D}_a \stackrel{\mathrm{def.}}{=} \hat{D}_{e_a^{\mathrm{cyl}}}\,, \quad \nabla^{\hat{S},\mathrm{cyl}}_a \stackrel{\mathrm{def.}}{=} \nabla^{\hat{S},\mathrm{cyl}}_{e_a^{\mathrm{cyl}}}\,, \quad \nabla^{\hat{S},\mathrm{cone}}_a \stackrel{\mathrm{def.}}{=} \nabla^{\hat{S},\mathrm{cone}}_{e_a^{\mathrm{cyl}}}\,,$$

(which are derivations of the $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-module $\Gamma(\hat{M},\hat{S})$) and:

$$A_a^{\mathrm{cyl}} \stackrel{\mathrm{def.}}{=} A^{\mathrm{cyl}}(e_a^{\mathrm{cyl}}) \in \Gamma(\hat{M}, \mathrm{End}(\hat{S}))\,, \quad A_a^{\mathrm{cone}} \stackrel{\mathrm{def.}}{=} A^{\mathrm{cone}}(e_a^{\mathrm{cyl}}) \in \Gamma(\hat{M}, \mathrm{End}(\hat{S}))\,.$$

Then:

$$\nabla^{\hat{S},\mathrm{cyl}} = \sum_{a=1}^{d+1} e^a_{\mathrm{cyl}} \otimes \nabla^{\hat{S},\mathrm{cyl}}_a\,, \quad \nabla^{\hat{S},\mathrm{cone}} = \sum_{a=1}^{d+1} e^a_{\mathrm{cyl}} \otimes \nabla^{\hat{S},\mathrm{cone}}_a\,, \quad \hat{D} = \sum_{a=1}^{d+1} e^a_{\mathrm{cyl}} \otimes \hat{D}_a$$

and:

$$A^{\mathrm{cyl}} = \sum_{m=1}^{d+1} e^a_{\mathrm{cyl}} \otimes A_a^{\mathrm{cyl}}\,, \quad A^{\mathrm{cone}} = \sum_{m=1}^{d+1} e^a_{\mathrm{cyl}} \otimes A_a^{\mathrm{cone}}\,,$$

where:

$$A^{\mathrm{cyl}}_{d+1} = 0\,, \quad A^{\mathrm{cyl}}_m = \Pi^*(A_m) \tag{2.74}$$

and:

$$A^{\mathrm{cone}}_{d+1} = 0\,, \tag{2.75}$$

$$A^{\mathrm{cone}}_m = A^{\mathrm{cyl}}_m - \frac{1}{2}\gamma_{\mathrm{cyl}}((e_m^{\mathrm{cyl}})_{\#_{\mathrm{cyl}}} \wedge \psi) = A^{\mathrm{cyl}}_m - \frac{1}{2r}\gamma_{\mathrm{cone}}((e_m^{\mathrm{cone}})_{\#_{\mathrm{cone}}} \wedge \theta) = A^{\mathrm{cyl}}_m - \frac{\epsilon}{2}\gamma_{*,m}\gamma_*^{(d+1)}$$

Notice that:

$$D^{\mathrm{cyl}}_{\partial_u} = \nabla^{\hat{S},\mathrm{cyl}}_{\partial_u} = \mathcal{L}^{\hat{S}}_{\partial_u}\,, \quad D^{\mathrm{cone}}_{\partial_r} = \nabla^{\hat{S},\mathrm{cone}}_{\partial_r} = \mathcal{L}^{\hat{S}}_{\partial_r}\,. \tag{2.76}$$

Cone and cylinder dequantizations of the lift of a general pin connection. As before, consider the lift $\hat{D} = D^*$ of a general linear connection $D = e^a \otimes D_a$ on S. Recall from [7] that the adjoint dequantization of D is given by:

$$\check{D}^{\rm ad}_m \omega \stackrel{\rm def.}{=} \nabla_m \omega + [\check{A}_m, \omega]_{-,\diamond}\,, \quad \forall \omega \in \Omega_{\mathbb{K}}(M)\,,$$

where:

$$\check{A}_m \stackrel{\rm def.}{=} \gamma^{-1}(A_m) \in \Omega_{\mathbb{K}}(M)\ .$$

As in [7], we define the *adjoint cylinder and cone dequantizations* of the lift $\hat{D}$ of D through:

$$(\check{D}^{\rm ad})^{\rm cyl} \stackrel{\rm def.}{=} \sum_{a=1}^{d+1} e^a_{\rm cyl} \otimes (\check{D}^{\rm ad}_a)^{\rm cyl}\,, \quad (\check{D}^{\rm ad})^{\rm cone} \stackrel{\rm def.}{=} \sum_{a=1}^{d+1} e^a_{\rm cyl} \otimes (\check{D}^{\rm ad}_a)^{\rm cone}\,,$$

where:

$$(\check{D}^{\rm ad}_a)^{\rm cyl}\omega \stackrel{\rm def.}{=} \nabla^{\rm cyl}_a\omega + [\check{A}^{\rm cyl}_a, \omega]_{-,\diamond^{\rm cyl}}\,,\ (\check{D}^{\rm ad}_a)^{\rm cone}\omega \stackrel{\rm def.}{=} \nabla^{\rm cone}_a\omega + [\check{A}^{\rm cone}_a, \omega]_{-,\diamond^{\rm cone}}\,,\ \forall \omega \in \Omega_{\mathbb{K}}(\hat{M})\,, \tag{2.77}$$

and:

$$\check{A}^{\rm cyl}_a \stackrel{\rm def.}{=} \gamma^{-1}_{\rm cyl}(A^{\rm cyl}_a) \in \Omega^{\epsilon,{\rm cyl}}_{\mathbb{K}}(\hat{M})\,, \quad \check{A}^{\rm cone}_a \stackrel{\rm def.}{=} \gamma^{-1}_{\rm cone}(A^{\rm cone}_a) \in \Omega^{\epsilon,{\rm cone}}_{\mathbb{K}}(\hat{M})\ . \tag{2.78}$$

Relations (2.51) and (2.74), (2.75) give:

$$\check{A}^{\rm cyl}_m = P^{\rm cyl}_\epsilon(\check{A}_{m,{\rm cyl}})\,, \quad \check{A}^{\rm cone}_m = P^{\rm cone}_\epsilon\left(\check{A}_{m,{\rm cone}} - \frac{1}{2}(e^m_{\rm cone})_{\#_{\rm cone}} \wedge \theta\right)\,, \quad \check{A}^{\rm cyl}_{d+1} = \check{A}^{\rm cone}_{d+1} = 0\,, \tag{2.79}$$

where:

$$\check{A}_{m,{\rm cyl}} \stackrel{\rm def.}{=} \Pi^*(\check{A}_m) \quad \text{and} \quad \check{A}_{m,{\rm cone}} \stackrel{\rm def.}{=} r^{\mathcal{E}}\Pi^*(\check{A}_m)$$

are the cylinder and cone lifts of the inhomogeneous differential forms $\check{A}_m$ (see equations (2.29) and (2.30)) and we used the identities $\gamma^{-1}_{\rm cyl}\circ\gamma_{\rm cyl} = P^{\rm cyl}_\epsilon$ and $\gamma^{-1}_{\rm cone}\circ\gamma_{\rm cone} = P^{\rm cone}_\epsilon$. In particular, we have:

$$\check{A}^{\rm cone}_m = r^{\mathcal{E}}(\check{A}^{\rm cyl}_m) - \frac{1}{2}P^{\rm cone}_\epsilon[(e^m_{\rm cone})_{\#_{\rm cone}} \wedge \theta]\,, \tag{2.80}$$

where we used relation (2.23). Equations (2.80), (2.77) and (2.42) imply that the two dequantized connections on $\hat{M}$ are related through:

$$(\check{D}^{\rm ad}_a)^{\rm cone} = r^{\mathcal{E}} \circ (\check{D}^{\rm ad}_a)^{\rm cyl} \circ r^{-\mathcal{E}}\ . \tag{2.81}$$

The pullback of $\check{D}^{\rm ad}$ gives an operator $(\check{D}^{\rm ad})^* : \Omega^{\perp,{\rm cyl}}_{\mathbb{K}}(\hat{M}) \to \Omega^1_{\mathbb{K}}(\hat{M}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \Omega^{\perp,{\rm cyl}}_{\mathbb{K}}(\hat{M})$ which expands as $(\check{D}^{\rm ad})^* = e^{\rm cyl}_a \otimes (\check{D}^{\rm ad}_a)^*$, where $(\check{D}^{\rm ad}_{d+1})^* = 0$ while $(\check{D}^{\rm ad}_m)^*$ are derivations of the algebra $(\Omega^{\perp,{\rm cyl}}_{\mathbb{K}}(\hat{M}), \diamond^{\rm cyl})$ which act as:

$$(\check{D}^{\rm ad}_m)^*\omega = \nabla^*_m\omega + [\Pi^*(\check{A}_m), \omega]_{-,\diamond^{\rm cyl}} = \nabla^{\rm cyl}_m\omega + [\check{A}_{m,{\rm cyl}}, \omega]_{-,\diamond^{\rm cyl}}\,,\ \forall \omega \in \Omega^{\perp,{\rm cyl}}_{\mathbb{K}}(\hat{M}) \approx \Pi^*(\Omega_{\mathbb{K}}(M))$$

and satisfy:

$$(\check{D}^{\rm ad}_m)^* \circ \Pi^* = \Pi^* \circ \check{D}^{\rm ad}_m\ . \tag{2.82}$$

Since $\nabla^{\rm cyl}$ commutes with $P_{\pm}^{\rm cyl}$, relations (2.79) imply:

$$(\check{D}_a^{\rm ad})^{\rm cyl}\circ P_\epsilon^{\rm cyl}=P_\epsilon^{\rm cyl}\circ(\check{D}_a^{\rm ad})^*\,,\quad (\check{D}_a^{\rm ad})^{\rm cone}\circ P_\epsilon^{\rm cone}\circ r^{\mathcal{E}}=P_\epsilon^{\rm cone}\circ r^{\mathcal{E}}\circ(\check{D}_a^{\rm ad})^*\,. \tag{2.83}$$

Together with (2.82), this gives the identities:

$$(\check{D}_a^{\rm ad})^{\rm cyl}\circ P_\epsilon^{\rm cyl}\circ\Pi^*=P_\epsilon^{\rm cyl}\circ\Pi^*\circ\check{D}_a^{\rm ad}\,,\quad (\check{D}_a^{\rm ad})^{\rm cone}\circ P_\epsilon^{\rm cone}\circ r^{\mathcal{E}}\circ\Pi^*=P_\epsilon^{\rm cone}\circ r^{\mathcal{E}}\circ\Pi^*\circ\check{D}_a^{\rm ad}\,, \tag{2.84}$$

which will be used later on.

Remark. The Clifford connection properties of $\nabla^{\hat{S},{\rm cyl}}$ and $\nabla^{\hat{S},{\rm cone}}$:

$$\gamma_{\rm cyl}\circ\nabla_a^{\rm cyl}=(\nabla_a^{\hat{S},{\rm cyl}})^{\rm ad}\circ\gamma_{\rm cyl}\,,\quad \gamma_{\rm cone}\circ\nabla_a^{\rm cone}=(\nabla_a^{\hat{S},{\rm cone}})^{\rm ad}\circ\gamma_{\rm cone}$$

imply:

$$\gamma_{\rm cyl}\circ(\check{D}_a^{\rm ad})^{\rm cyl}=\hat{D}_a^{\rm ad}\circ\gamma_{\rm cyl}\,,\quad \gamma_{\rm cone}\circ(\check{D}_a^{\rm ad})^{\rm cone}=\hat{D}_a^{\rm ad}\circ\gamma_{\rm cone}\,,$$

i.e.:

$$P_\epsilon^{\rm cyl}\circ(\check{D}_a^{\rm ad})^{\rm cyl}=\gamma_{\rm cyl}^{-1}\circ\hat{D}_a^{\rm ad}\circ\gamma_{\rm cyl}\,,\quad P_\epsilon^{\rm cone}\circ(\check{D}_a^{\rm ad})^{\rm cone}=\gamma_{\rm cone}^{-1}\circ\hat{D}_a^{\rm ad}\circ\gamma_{\rm cone}\,.$$

2.8 Lifting algebraic constraints on pinors. Dequantizations of lifted algebraic constraints

The lift of algebraic constraints. Given an endomorphism $Q\in\Gamma(M,\mathrm{End}(S))$, we define its lift $\hat{Q}$ to $\hat{M}$ to be the pullback of Q to a globally-defined endomorphism of the pin bundle $\hat{S}$ of $\hat{M}$:

$$\hat{Q}\stackrel{\rm def.}{=}\Pi^*(Q)\in\Gamma(\hat{M},\mathrm{End}(\hat{S}))\,. \tag{2.85}$$

Since $\hat{Q}\circ\Pi^*=\Pi^*\circ Q$, the pullback map (2.61) induces an injective but non-surjective $\mathcal{C}^\infty(M,\mathbb{K})$-linear map from the space $\mathcal{K}(Q)=\{\xi\in\Gamma(M,S)|Q\xi=0\}$ to the space $\mathcal{K}(\hat{Q})=\{\hat{\xi}\in\Gamma(\hat{M},\hat{S})|\hat{Q}\hat{\xi}=0\}$:

$$\mathcal{K}(Q)\stackrel{\Pi^*|_{\mathcal{K}(Q)}}{\longrightarrow}\mathcal{K}(\hat{Q})\,,$$

where, as usual, we identify $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})\approx\mathcal{C}^\infty(M,\mathbb{K})$. The image of this map is the following $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-submodule of $\Gamma(\hat{M},\hat{S})$:

$$\Pi^*(\mathcal{K}(Q))=\mathcal{K}(\hat{Q})\cap\Gamma^{\rm vert}(\hat{M},\hat{S})\stackrel{\rm def.}{=}\mathcal{K}^{\rm vert}(\hat{Q})\,,$$

which we shall call the $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-*module of vertical* $\hat{Q}$-*constrained pinors on* $\hat{M}$. Identifying $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})\approx\mathcal{C}^\infty(M,\mathbb{K})$, the appropriate restrictions of Π^* and j^* give mutually-inverse isomorphisms of $\mathcal{C}^\infty(M,\mathbb{K})$-modules:

$$\mathcal{K}(Q)\quad\underset{j^*|^{\mathcal{K}(Q)}_{\mathcal{K}^{\rm vert}(\hat{Q})}}{\overset{\Pi^*|^{\mathcal{K}^{\rm vert}(\hat{Q})}_{\mathcal{K}(Q)}}{\rightleftarrows}}\quad\mathcal{K}^{\rm vert}(\hat{Q})\,. \tag{2.86}$$

Hence a pinor $\hat{\xi}\in\Gamma(\hat{M},\hat{S})$ on $\hat{M}$ satisfies $\hat{Q}\hat{\xi}=\mathcal{L}^{\hat{S}}_{\partial_r}\hat{\xi}=0$ iff. it is the pullback $\hat{\xi}=\Pi^*(\xi)$ of a pinor $\xi\in\Gamma(M,S)$ which satisfies $Q\xi=0$. This allows one to translate between algebraic constraints on pinors defined on M and on pinors defined on $\hat{M}$.

Cone and cylinder dequantizations of the lift of an algebraic constraint. As in [7], consider the dequantization

$$\check{Q} \stackrel{\text{def.}}{=} \gamma^{-1}(Q) \in \Omega_{\mathbb{K}}(M)$$

of $Q \in \mathrm{End}(S)$ as well as the *cone and cylinder dequantizations* of the lift $\hat{Q}$ of Q:

$$\begin{aligned} \check{Q}^{\mathrm{cyl}} &\stackrel{\text{def.}}{=} \gamma_{\mathrm{cyl}}^{-1}(\hat{Q}) = (P_\epsilon^{\mathrm{cyl}} \circ \Pi^*)(\check{Q}) = P_\epsilon^{\mathrm{cyl}}(\check{Q}_{\mathrm{cyl}}) \in \Omega_{\mathbb{K}}^{\epsilon,\mathrm{cyl}}(\hat{M})\,, \\ \check{Q}^{\mathrm{cone}} &\stackrel{\text{def.}}{=} \gamma_{\mathrm{cone}}^{-1}(\hat{Q}) = (P_\epsilon^{\mathrm{cone}} \circ r^{\mathcal{E}} \circ \Pi^*)(\check{Q}) = P_\epsilon^{\mathrm{cone}}(\check{Q}_{\mathrm{cone}}) = r^{\mathcal{E}}(\check{Q}^{\mathrm{cyl}}) \in \Omega_{\mathbb{K}}^{\epsilon,\mathrm{cone}}(\hat{M})\,, \end{aligned} \tag{2.87}$$

where we used relations (2.51) and where

$$\check{Q}_{\mathrm{cyl}} \stackrel{\text{def.}}{=} \Pi^*(\check{Q}) \in \Omega_{\mathbb{K}}^{\mathrm{cyl}}(\hat{M})\,, \quad \check{Q}_{\mathrm{cone}} = r^{\mathcal{E}}(\Pi^*(\check{Q})) = r^{\mathcal{E}}(\check{Q}_{\mathrm{cyl}}) \in \Omega_{\mathbb{K}}^{\mathrm{cone}}(\hat{M})$$

are the cylinder and cone lifts of the inhomogeneous form $\check{Q} \in \Omega_{\mathbb{K}}(M)$ (see relations (2.29) and (2.30)).

Lifting the algebra of constrained differential forms. As in [7], consider the $\mathcal{C}^\infty(M,\mathbb{K})$-algebra of constrained inhomogeneous forms on M:

$$\check{\mathcal{K}}_Q \stackrel{\text{def.}}{=} \check{\mathcal{K}}(Q) = \check{E}(\mathcal{K}(\hat{Q}) \otimes_{\mathcal{C}^\infty(M,\mathbb{K})} \mathcal{K}(Q)) = \mathcal{K}(L_{\check{Q}}) \cap \mathcal{K}(R_{\tau_{\mathscr{B}}(\check{Q})})$$

as well as the $\mathcal{C}^\infty(\hat{M},\mathbb{K})$-algebras of constrained inhomogeneous forms on the cylinder and cone:

$$\begin{aligned} \check{\mathcal{K}}_{\hat{Q},\mathrm{cyl}} &\stackrel{\text{def.}}{=} \check{\mathcal{K}}^{\mathrm{cyl}}(\hat{Q}) = \check{E}^{\mathrm{cyl}}(\mathcal{K}(\hat{Q}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \mathcal{K}(\hat{Q})) = \mathcal{K}(L^{\mathrm{cyl}}_{\check{Q}^{\mathrm{cyl}}}) \cap \mathcal{K}(R^{\mathrm{cyl}}_{\tau_{\hat{\mathscr{B}}}(\check{Q}^{\mathrm{cyl}})}) \cap \Omega^{\epsilon}_{\mathbb{K},\mathrm{cyl}}(\hat{M}) \\ \check{\mathcal{K}}_{\hat{Q},\mathrm{cone}} &\stackrel{\text{def.}}{=} \check{\mathcal{K}}^{\mathrm{cone}}(\hat{Q}) = \check{E}^{\mathrm{cone}}(\mathcal{K}(\hat{Q}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \mathcal{K}(\hat{Q})) \\ &\qquad\qquad = \mathcal{K}(L^{\mathrm{cone}}_{\check{Q}^{\mathrm{cone}}}) \cap \mathcal{K}(R^{\mathrm{cone}}_{\tau_{\hat{\mathscr{B}}}(\check{Q}^{\mathrm{cone}})}) \cap \Omega^{\epsilon}_{\mathbb{K},\mathrm{cone}}(\hat{M})\,. \end{aligned} \tag{2.88}$$

We also define the $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebras of *special constrained inhomogeneous forms* on the cylinder and cone through:

$$\begin{aligned} \check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{Q}} &\stackrel{\text{def.}}{=} \check{\mathcal{K}}_{\hat{Q},\mathrm{cyl}} \cap \Omega_{\mathbb{K}}^{\mathrm{cyl}}(\hat{M}) = \mathcal{K}(L^{\mathrm{cyl}}_{\check{Q}^{\mathrm{cyl}}}) \cap \mathcal{K}(R^{\mathrm{cyl}}_{\tau_{\hat{\mathscr{B}}}(\check{Q}^{\mathrm{cyl}})}) \cap \Omega_{\mathbb{K}}^{\epsilon,\mathrm{cyl}}(\hat{M}) \\ \check{\mathcal{K}}^{\mathrm{cone}}_{\hat{Q}} &\stackrel{\text{def.}}{=} \check{\mathcal{K}}_{\hat{Q},\mathrm{cone}} \cap \Omega_{\mathbb{K}}^{\mathrm{cone}}(\hat{M}) = \mathcal{K}(L^{\mathrm{cone}}_{\check{Q}^{\mathrm{cone}}}) \cap \mathcal{K}(R^{\mathrm{cone}}_{\tau_{\hat{\mathscr{B}}}(\check{Q}^{\mathrm{cone}})}) \cap \Omega_{\mathbb{K}}^{\epsilon,\mathrm{cone}}(\hat{M})\,. \end{aligned} \tag{2.89}$$

The relation $\check{Q}^{\mathrm{cone}} = r^{\mathcal{E}}(\check{Q}^{\mathrm{cyl}})$ and identities (2.14) imply:

$$\mathcal{K}(L^{\mathrm{cone}}_{\check{Q}^{\mathrm{cone}}}) = r^{\mathcal{E}}(\mathcal{K}(L^{\mathrm{cyl}}_{\check{Q}^{\mathrm{cyl}}}))\,, \quad \mathcal{K}(R^{\mathrm{cone}}_{\tau_{\check{\mathscr{B}}}(\check{Q}^{\mathrm{cone}})}) = r^{\mathcal{E}}(\mathcal{K}(R^{\mathrm{cyl}}_{\tau_{\check{\mathscr{B}}}(\check{Q}^{\mathrm{cyl}})}))\,.$$

Together with $r^{\mathcal{E}}(\Omega_{\mathbb{K}}^{\epsilon,\mathrm{cyl}}(\hat{M})) = \Omega_{\mathbb{K}}^{\epsilon,\mathrm{cone}}(\hat{M})$, this gives:

$$\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{Q}} = r^{\mathcal{E}}(\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{Q}})\,.$$

Relations (2.60) imply:

$$\begin{aligned} \check{\mathcal{K}}_{\hat{Q},\mathrm{cyl}} &= P_\epsilon^{\mathrm{cyl}}(\check{\mathcal{K}}_{\hat{Q}})\,, & \check{\mathcal{K}}_{\hat{Q},\mathrm{cone}} &= (P_\epsilon^{\mathrm{cone}} \circ r^{\mathcal{E}})(\check{\mathcal{K}}_{\hat{Q}})\,, \\ \check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{Q}} &= P_\epsilon^{\mathrm{cyl}}(\check{\mathcal{K}}^{\mathrm{vert}}_{\hat{Q}})\,, & \check{\mathcal{K}}^{\mathrm{cone}}_{\hat{Q}} &= (P_\epsilon^{\mathrm{cone}} \circ r^{\mathcal{E}})(\check{\mathcal{K}}^{\mathrm{vert}}_{\hat{Q}})\,, \end{aligned}$$

where we defined:

$$\check{\mathcal{K}}_{\hat{Q}} \stackrel{\text{def.}}{=} \check{E}_*(\mathcal{K}(\hat{Q}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \mathcal{K}(\hat{Q})) \subset \Omega^\perp_{\mathbb{K}}(\hat{M}) \,,$$
$$\check{\mathcal{K}}^{\text{vert}}_{\hat{Q}} \stackrel{\text{def.}}{=} \check{E}_*(\mathcal{K}^{\text{vert}}(\hat{Q}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \mathcal{K}^{\text{vert}}(\hat{Q})) \subset \Omega^{\perp,\text{cyl}}_{\mathbb{K}}(\hat{M}) \,.$$

Relation (2.58) implies:

$$\check{\mathcal{K}}^{\text{vert}}_{\hat{Q}} = \Pi^*(\check{\mathcal{K}}(Q)) \,, \tag{2.90}$$

which in turn gives:

$$\check{\mathcal{K}}^{\text{cyl}}_{\hat{Q}} = (P^{\text{cyl}}_\epsilon \circ \Pi^*)(\check{\mathcal{K}}_Q) \,, \quad \check{\mathcal{K}}^{\text{cone}}_{\hat{Q}} = (P^{\text{cyl}}_\epsilon \circ r^{\mathcal{E}} \circ \Pi^*)(\check{\mathcal{K}}_Q) \,.$$

Hence the isomorphisms of $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K}) \approx \mathcal{C}^\infty(M,\mathbb{K})$-algebras:

$$P^{\text{cyl}}_\epsilon \circ \Pi^* : (\Omega_{\mathbb{K}}(M), \diamond) \xrightarrow{\sim} (\Omega^{\epsilon,\text{cyl}}_{\mathbb{K}}(\hat{M}), \diamond^{\text{cyl}}) \,, \quad P^{\text{cone}}_\epsilon \circ r^{\mathcal{E}} \circ \Pi^* : (\Omega_{\mathbb{K}}(M), \diamond) \xrightarrow{\sim} (\Omega^{\epsilon,\text{cone}}_{\mathbb{K}}(\hat{M}), \diamond^{\text{cone}}) \tag{2.91}$$

restrict to isomorphism of $\mathcal{C}^\infty(M,\mathbb{K})$-algebras from $\mathcal{K}_Q$ to $\check{\mathcal{K}}^{\text{cyl}}_{\hat{Q}}$ and $\check{\mathcal{K}}^{\text{cone}}_{\hat{Q}}$, respectively, whose inverses are given by the appropriate restrictions of $j^* \circ P_\perp$. It follows that the $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebras $(\check{\mathcal{K}}^{\text{cyl}}_{\hat{Q}}, \diamond^{\text{cyl}})$ and $(\check{\mathcal{K}}^{\text{cone}}_{\hat{Q}}, \diamond^{\text{cone}})$ give models for the $\mathcal{C}^\infty(M,\mathbb{K})$-algebra $(\check{\mathcal{K}}_Q, \diamond)$, allowing to translate between constrained inhomogeneous differential forms defined on M and those defined on the cylinder and cone over M.

$$\begin{tikzcd}
(\check{\mathcal{K}}_Q, \diamond) \arrow[r, shift left, "\Pi^*"] \arrow[d, shift left] & (\check{\mathcal{K}}^{\text{vert}}_{\hat{Q}}, \diamond^{\text{cyl}}) \arrow[l, shift left, "j^*"] \arrow[d, shift left, "P^{\text{cyl}}_\epsilon"] \\
(\check{\mathcal{K}}^{\text{cone}}_{\hat{Q}}, \diamond^{\text{cone}}) \arrow[u, shift left] \arrow[r, shift left, "r^{-\mathcal{E}}"] & (\check{\mathcal{K}}^{\text{cyl}}_{\hat{Q}}, \diamond^{\text{cyl}}) \arrow[l, shift left, "r^{\mathcal{E}}"] \arrow[u, shift left, "P_\perp"]
\end{tikzcd} \tag{2.92}$$

2.9 Lifting generalized Killing pinors and generalized Killing forms

The lift of generalized Killing pinors. Relations (2.76) imply that the Killing pinor equations with respect to $\hat{D}$ for a pinor $\hat{\xi} \in \Gamma(\hat{M}, \hat{S})$ amount to the condition $\mathcal{L}^{\hat{S}}_{\partial_u}\hat{\xi} = 0 \Leftrightarrow \mathcal{L}^{\hat{S}}_{\partial_r}\hat{\xi} = 0$ (which says that $\hat{\xi}$ coincides with the pullback $\xi_* = \Pi^*(\xi)$ of some pinor $\xi \in \Gamma(M,S)$ defined on M) together with the conditions $(\nabla^S_m)^*\hat{\xi} + (A_m)_*\hat{\xi} = 0 \Leftrightarrow \Pi^*(D_m\xi) = 0$ (which are equivalent with the generalized Killing pinor equations $D_m\xi = 0$ on M). In particular, the space:

$$\mathcal{K}(\hat{D}) \stackrel{\text{def.}}{=} \cap_{a=1}^{d+1}\mathcal{K}(\hat{D}_a) = \{\hat{\xi} \in \Gamma(\hat{M},\hat{S}) | \hat{D}_a\xi = 0 \,, \ \ \forall a = 1, \ldots, d+1\}$$

is a subspace of $\Gamma^{\text{vert}}(\hat{M},\hat{S})$ which coincides with the Π-pullback of the space:

$$\mathcal{K}(D) \stackrel{\text{def.}}{=} \cap_{m=1}^{d}\mathcal{K}(D_m) = \{\xi \in \Gamma(M,S) | D_m\xi = 0 \,, \ \ \forall m = 1, \ldots, d\} \,.$$

The relation:

$$\mathcal{K}(\hat{D}) = \Pi^*(\mathcal{K}(D))$$

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shows that Π^* induces a isomorphism of $\mathbb{K}$-vector spaces between $\mathcal{K}(D)$ and $\mathcal{K}(\hat{D})$, whose inverse is given by the appropriate restriction of j^*:

$$\mathcal{K}(D) \underset{j^*|_{\mathcal{K}(\hat{D})}}{\overset{\Pi^*|_{\mathcal{K}(D)}}{\rightleftarrows}} \mathcal{K}(\hat{D})\,. \tag{2.93}$$

This allows us to translate between generalized Killing pinor equations on M and $\hat{M}$.

Lifting the flat Fierz $\mathbb{K}$-algebra of generalized Killing pinors. As in [7], consider the flat Fierz $\mathbb{K}$-algebra of D on M:

$$\check{\mathcal{K}}(D) \stackrel{\text{def.}}{=} \check{E}(\mathcal{K}(D) \otimes_{\mathcal{C}^\infty(M,\mathbb{K})} \mathcal{K}(D)) \subset \Omega_{\mathbb{K}}(M)\,,$$

which is a $\mathbb{K}$-subalgebra of the Kähler-Atiyah algebra $(\Omega_{\mathbb{K}}(M), \diamond)$. Let us define:

$$\check{\mathcal{K}}(\hat{D}) \stackrel{\text{def.}}{=} \check{E}_*(\mathcal{K}(\hat{D}) \otimes_{\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})} \mathcal{K}(\hat{D})) \subset \Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M})\,,$$

which is a $\mathbb{K}$-subalgebra of the $\mathcal{C}^\infty_\perp(\hat{M},\mathbb{K})$-algebra $(\Omega_{\mathbb{K}}^{\perp,\mathrm{cyl}}(\hat{M}), \diamond^{\mathrm{cyl}})$. Relations (2.57) imply:

$$\check{\mathcal{K}}(\hat{D}) = \Pi^*(\check{\mathcal{K}}(D))\,,$$

so the appropriate co-restriction of Π^* gives a unital isomorphism of $\mathbb{K}$-algebras from $(\check{\mathcal{K}}(D), \diamond)$ to $(\check{\mathcal{K}}(\hat{D}), \diamond^{\mathrm{cyl}})$. As in [7], consider now the flat Fierz $\mathbb{K}$-algebras determined on the cylinder and cone by the $\hat{\mathscr{B}}$-flat subspace $\mathcal{K}(\hat{D}) \subset \Gamma(\hat{M}, \hat{S})$:

$$\begin{aligned}
\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}) &\stackrel{\text{def.}}{=} \check{E}^{\mathrm{cyl}}(\mathcal{K}(\hat{D}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \mathcal{K}(\hat{D})) \subset \Omega_{\mathbb{K}}^{\epsilon,\mathrm{cyl}}(\hat{M})\,,\\
\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D}) &\stackrel{\text{def.}}{=} \check{E}^{\mathrm{cone}}(\mathcal{K}(\hat{D}) \otimes_{\mathcal{C}^\infty(\hat{M},\mathbb{K})} \mathcal{K}(\hat{D})) \subset \Omega_{\mathbb{K}}^{\epsilon,\mathrm{cyl}}(\hat{M})\,.
\end{aligned}$$

Relations (2.60) imply:

$$\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}) = P_\epsilon^{\mathrm{cyl}}(\check{\mathcal{K}}(\hat{D}))\,,\quad \check{\mathcal{K}}^{\mathrm{cone}}(\hat{D}) = (P_\epsilon^{\mathrm{cone}} \circ r^{\mathcal{E}})(\check{\mathcal{K}}(\hat{D})) = r^{\mathcal{E}}(\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}))\,. \tag{2.94}$$

When combined with (2.9), the last equations show that the appropriate restrictions of the morphisms of algebras (2.91) give isomorphisms of algebras between $\check{\mathcal{K}}(D)$ and $\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D})$, respectively $\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D})$, whose inverses are given by $j^* \circ P_\perp$ and $j^* \circ r^{-\mathcal{E}} \circ P_\perp$, respectively. The situation is summarized in the commutative diagram:

$$\begin{tikzcd}
(\check{\mathcal{K}}(D), \diamond) \arrow[r, shift left, "\Pi^*"] \arrow[d, shift left] & (\check{\mathcal{K}}(\hat{D}), \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "j^*"] \arrow[d, shift left, "P_\epsilon^{\mathrm{cyl}}"] \\
(\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D}), \diamond^{\mathrm{cone}}) \arrow[u, shift left] \arrow[r, shift left, "r^{-\mathcal{E}}"] & (\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}), \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "r^{\mathcal{E}}"] \arrow[u, shift left, "P_\perp"]
\end{tikzcd} \tag{2.95}$$

The lift of generalized Killing forms. As in [7], consider the generalized Killing $\mathbb{K}$-algebra determined by D on M:

$$\check{\mathcal{K}}_D \stackrel{\text{def.}}{=} \mathcal{K}(\check{D}^{\mathrm{ad}}) \subset \Omega_{\mathbb{K}}(M)$$

as well as the generalized Killing $\mathbb{K}$-algebras determined by $\hat{D}$ on the cylinder and cone:

$$\begin{aligned}
\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D}} &\stackrel{\text{def.}}{=} \mathcal{K}((\check{D}^{\mathrm{ad}})^{\mathrm{cyl}}) \cap \Omega^{\epsilon}_{\mathbb{K},\mathrm{cyl}}(\hat{M}) = \mathcal{K}((\check{D}^{\mathrm{ad}})^{\mathrm{cyl}}) \cap \Omega^{\epsilon,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})\,, \\
\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D}} &\stackrel{\text{def.}}{=} \mathcal{K}((\check{D}^{\mathrm{ad}})^{\mathrm{cone}}) \cap \Omega^{\epsilon}_{\mathbb{K},\mathrm{cone}}(\hat{M}) = \mathcal{K}((\check{D}^{\mathrm{ad}})^{\mathrm{cone}}) \cap \Omega^{\epsilon,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})\,,
\end{aligned}$$

where we noticed that $\mathcal{K}((\check{D}^{\mathrm{ad}})^{\mathrm{cyl}}) \subset \Omega^{\mathrm{cyl}}_{\mathbb{K}}(\hat{M})$ and $\mathcal{K}((\check{D}^{\mathrm{ad}})^{\mathrm{cone}}) \subset \Omega^{\mathrm{cone}}_{\mathbb{K}}(\hat{M})$. Also consider the following subalgebra of $(\Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}), \diamond^{\mathrm{cyl}})$:

$$\check{\mathcal{K}}_{\hat{D}} \stackrel{\text{def.}}{=} \mathcal{K}((\check{D}^{\mathrm{ad}})^*) = \cap_{m=1}^{d} \mathcal{K}((\check{D}^{\mathrm{ad}}_m)^*) \subset \Omega^{\perp,\mathrm{cyl}}_{\mathbb{K}}(\hat{M})\ .$$

Relation (2.82) implies:

$$\check{\mathcal{K}}_{\hat{D}} = \Pi^*(\check{\mathcal{K}}_D)\ . \tag{2.96}$$

Relations (2.60) give:

$$\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D}} = P^{\mathrm{cyl}}_{\epsilon}(\check{\mathcal{K}}_{\hat{D}})\,,\quad \check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D}} = (P^{\mathrm{cone}}_{\epsilon} \circ r^{\mathcal{E}})(\check{\mathcal{K}}_{\hat{D}}) = r^{\mathcal{E}}(\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D}})\ . \tag{2.97}$$

Combining this with (2.96), we find that the morphisms of algebras (2.91) restrict to isomorphisms of $\mathbb{K}$-algebras between $\check{\mathcal{K}}_D$ and $\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D}}$, respectively $\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D}}$. The situation is summarized in the commutative diagram:

$$\begin{tikzcd}
(\check{\mathcal{K}}_D, \diamond) \arrow[r, shift left, "\Pi^*"] \arrow[d, shift left] & (\check{\mathcal{K}}_{\hat{D}}, \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "j^*"] \arrow[d, shift left, "P^{\mathrm{cyl}}_{\epsilon}"] \\
(\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D}}, \diamond^{\mathrm{cone}}) \arrow[u, shift left] \arrow[r, shift left, "r^{-\mathcal{E}}"] & (\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D}}, \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "r^{\mathcal{E}}"] \arrow[u, shift left, "P_{\perp}"]
\end{tikzcd}\ . \tag{2.98}$$

2.10 Relating CGK pinors and CGK forms on M and $\hat{M}$

Lifting the CGK pinor equations from M to $\hat{M}$. Consider the CGK pinor equations defined on M by some connection D on S and by a single endomorphism $Q \in \Gamma(M, \mathrm{End}(S))$ as well as the CGK pinor equations defined on $\hat{M}$ by the lifts $\hat{D}$ and $\hat{Q}$, as before. Denoting the corresponding $\mathbb{K}$-vector spaces of solutions by $\mathcal{K}(D,Q) = \mathcal{K}(D) \cap \mathcal{K}(Q)$ and $\mathcal{K}(\hat{D},\hat{Q}) = \mathcal{K}(\hat{D}) \cap \mathcal{K}(\hat{Q})$, the observations of the previous subsections imply that we have the inclusion:

$$\mathcal{K}(\hat{D},\hat{Q}) \subset \Gamma^{\mathrm{vert}}(\hat{M},\hat{S})$$

and that we have mutually-inverse isomorphisms of $\mathbb{K}$-vector spaces:

$$\mathcal{K}(D,Q) \underset{j^*|_{\mathcal{K}(\hat{D},\hat{Q})}}{\overset{\Pi^*|_{\mathcal{K}(D)}}{\rightleftarrows}} \mathcal{K}(\hat{D},\hat{Q})\ . \tag{2.99}$$

This observation allows us to lift the CGK pinor equations from M to $\hat{M}$.

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Lifting CGK forms to the cylinder and cone. When using the cylinder or cone metric on $\hat{M}$, we can of course apply the formalism of [7] on $\hat{M}$, thereby working with the dequantized connections $(\check{D}^{\mathrm{ad}})^{\mathrm{cyl}}$ and $(\check{D}^{\mathrm{ad}})^{\mathrm{cone}}$ which were discussed above. Combining the observations of the previous subsections shows that the $\mathbb{K}$-algebras of CGK forms on the cylinder and cone are related to the $\mathbb{K}$-algebra of CGK forms on (M, g) through:

$$\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D},\hat{Q}} = (P^{\mathrm{cyl}}_\epsilon \circ \Pi^*)(\check{\mathcal{K}}_{D,Q}) \,, \quad \check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D},\hat{Q}} = (P^{\mathrm{cone}}_\epsilon \circ r^{\mathcal{E}} \circ \Pi^*)(\check{\mathcal{K}}_{D,Q}) = r^{\mathcal{E}}(\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D},\hat{Q}})$$

while the flat Fierz $\mathbb{K}$-algebras determined on the cylinder and cone by the $\hat{\mathscr{B}}$-flat subspace $\mathcal{K}(\hat{D}, \hat{Q}) \subset \Gamma(\hat{M}, \hat{S})$ are related to the flat Fierz $\mathbb{K}$-algebra determined on M by the $\mathscr{B}$-flat subspace $\mathcal{K}(D, Q) \subset \Gamma(M, S)$ through:

$$\begin{aligned} \check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}, \hat{Q}) &= (P^{\mathrm{cyl}}_\epsilon \circ \Pi^*)(\check{\mathcal{K}}(D, Q)) \,, \\ \check{\mathcal{K}}^{\mathrm{cone}}(\hat{D}, \hat{Q}) &= (P^{\mathrm{cone}}_\epsilon \circ r^{\mathcal{E}} \circ \Pi^*)(\check{\mathcal{K}}(D, Q)) = r^{\mathcal{E}}(\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}, \hat{Q})) \,. \end{aligned}$$

Since $P^{\mathrm{cyl}}_\epsilon \circ \Pi^*$ and $P^{\mathrm{cone}}_\epsilon \circ r^{\mathcal{E}} \circ \Pi^*$ are isomorphism of $\mathbb{K}$-algebras, the relations above show that $(\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cyl}})$, $(\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cone}})$ and $(\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}, \hat{Q}), \diamond^{\mathrm{cyl}})$, $(\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D}, \hat{Q}), \diamond^{\mathrm{cone}})$ provide isomorphic models of $\check{\mathcal{K}}_D$ and $\check{\mathcal{K}}(D)$, respectively. The situation is summarized in the commutative diagrams:

$$\begin{tikzcd} (\check{\mathcal{K}}_{D,Q}, \diamond) \arrow[r, shift left, "\Pi^*"] \arrow[d, shift left] & (\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "j^*"] \arrow[d, shift left, "P^{\mathrm{cyl}}_\epsilon"] \\ (\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cone}}) \arrow[u, shift left] \arrow[r, shift left, "r^{-\mathcal{E}}"] & (\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "r^{\mathcal{E}}"] \arrow[u, shift left, "P_\perp"] \end{tikzcd} \qquad \begin{tikzcd} (\check{\mathcal{K}}(D, Q), \diamond) \arrow[r, shift left, "\Pi^*"] \arrow[d, shift left] & (\check{\mathcal{K}}(\hat{D}, \hat{Q}), \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "j^*"] \arrow[d, shift left, "P^{\mathrm{cyl}}_\epsilon"] \\ (\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D}, \hat{Q}), \diamond^{\mathrm{cone}}) \arrow[u, shift left] \arrow[r, shift left, "r^{-\mathcal{E}}"] & (\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}, \hat{Q}), \diamond^{\mathrm{cyl}}) \arrow[l, shift left, "r^{\mathcal{E}}"] \arrow[u, shift left, "P_\perp"] \end{tikzcd} \tag{2.100}$$

2.11 Lifting truncated models

Other isomorphic models — which are particularly convenient for computer computations — are obtained upon applying the isomorphisms of algebras given in section 3 of [7]. To describe this, we define:

$$\begin{aligned} \check{Q}^{<,\mathrm{cyl}} &\stackrel{\mathrm{def.}}{=} P_<(\check{Q}^{\mathrm{cyl}}) \in \Omega^{<,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \,, & \check{Q}^{<,\mathrm{cone}} &\stackrel{\mathrm{def.}}{=} P_<(\check{Q}^{\mathrm{cone}}) \in \Omega^{<,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \,, \\ \check{A}^{<,\mathrm{cyl}}_a &\stackrel{\mathrm{def.}}{=} P_<(\check{A}^{\mathrm{cyl}}_a) \in \Omega^{<,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}) \,, & \check{A}^{<,\mathrm{cone}}_a &\stackrel{\mathrm{def.}}{=} P_<(\check{A}^{\mathrm{cone}}_a) \in \Omega^{<,\mathrm{cone}}_{\mathbb{K}}(\hat{M}) \,, \\ D^{\mathrm{ad},<,\mathrm{cyl}}_a &\stackrel{\mathrm{def.}}{=} \nabla^{\mathrm{cyl}}_a + 2[A^{<,\mathrm{cyl}}_a, \,]_{-,\blacklozenge^{\mathrm{cyl}}_\epsilon} \,, & D^{\mathrm{ad},<,\mathrm{cone}}_a &\stackrel{\mathrm{def.}}{=} \nabla^{\mathrm{cone}}_a + 2[A^{<,\mathrm{cone}}_a, \,]_{-,\blacklozenge^{\mathrm{cone}}_\epsilon} \end{aligned}$$

as well as:

$$\begin{aligned} \check{\mathcal{K}}^{<,\mathrm{cyl}}_{\hat{D},\hat{Q}} &\stackrel{\mathrm{def.}}{=} P_<(\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D},\hat{Q}}) \,, & \check{\mathcal{K}}^{<,\mathrm{cone}}_{\hat{D},\hat{Q}} &\stackrel{\mathrm{def.}}{=} P_<(\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D},\hat{Q}}) \,, \\ \check{\mathcal{K}}^{<,\mathrm{cyl}}(\hat{D}, \hat{Q}) &\stackrel{\mathrm{def.}}{=} P_<(\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D}, \hat{Q})) \,, & \check{\mathcal{K}}^{<,\mathrm{cone}}(\hat{D}, \hat{Q}) &\stackrel{\mathrm{def.}}{=} P_<(\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D}, \hat{Q})) \,. \end{aligned}$$

Using the results above, it is not hard to see that the isomorphisms of algebras (see diagrams (2.34) and (2.35)):

$$\begin{aligned} (\Xi^{\mathrm{cyl}}_\epsilon)^{-1} &\stackrel{\mathrm{def.}}{=} 2P_< \circ P^{\mathrm{cyl}}_\epsilon \circ \Pi^* : (\Omega_{\mathbb{K}}(M), \diamond) \xrightarrow{\sim} (\Omega^{<,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}), \blacklozenge^{\mathrm{cyl}}_\epsilon) \,, \\ (\Xi^{\mathrm{cone}}_\epsilon)^{-1} &\stackrel{\mathrm{def.}}{=} 2P_< \circ P^{\mathrm{cone}}_\epsilon \circ r^{\mathcal{E}} \circ \Pi^* : (\Omega_{\mathbb{K}}(M), \diamond) \xrightarrow{\sim} (\Omega^{<,\mathrm{cyl}}_{\mathbb{K}}(\hat{M}), \blacklozenge^{\mathrm{cyl}}_\epsilon) \end{aligned}$$

satisfy:

$$
\begin{aligned}
2L^{\mathrm{cyl}}_{\check{Q}^{<,\mathrm{cyl}}} \circ (\Xi^{\mathrm{cyl}}_\epsilon)^{-1} &= (\Xi^{\mathrm{cyl}}_\epsilon)^{-1} \circ L_{\check{Q}}\,, \qquad 2R^{\mathrm{cyl}}_{\tau_{\hat{\mathscr{B}}}(\check{Q}^{<,\mathrm{cyl}})} \circ (\Xi^{\mathrm{cyl}}_\epsilon)^{-1} = (\Xi^{\mathrm{cyl}}_\epsilon)^{-1} \circ R_{\tau_{\mathscr{B}}(\check{Q})}\,,\\
2L^{\mathrm{cone}}_{\check{Q}^{<,\mathrm{cone}}} \circ (\Xi^{\mathrm{cone}}_\epsilon)^{-1} &= (\Xi^{\mathrm{cone}}_\epsilon)^{-1} \circ L_{\check{Q}}\,, \quad 2R^{\mathrm{cone}}_{\tau_{\hat{\mathscr{B}}}(\check{Q}^{<,\mathrm{cone}})} \circ (\Xi^{\mathrm{cone}}_\epsilon)^{-1} = (\Xi^{\mathrm{cone}}_\epsilon)^{-1} \circ R_{\tau_{\mathscr{B}}(\check{Q})}
\end{aligned}
$$

as well as:

$$
D^{\mathrm{ad},<,\mathrm{cyl}}_m \circ (\Xi^{\mathrm{cyl}}_\epsilon)^{-1} = (\Xi^{\mathrm{cyl}}_\epsilon)^{-1} \circ D^{\mathrm{ad}}_m \;, \quad D^{\mathrm{ad},<,\mathrm{cone}}_m \circ (\Xi^{\mathrm{cone}}_\epsilon)^{-1} = (\Xi^{\mathrm{cone}}_\epsilon)^{-1} \circ D^{\mathrm{ad}}_m \;.
$$

Together with the discussion of the previous subsections, this implies:

$$
\begin{aligned}
\check{\mathcal{K}}^{<,\mathrm{cyl}}_{\hat{D},\hat{Q}} &= (\Xi^{\mathrm{cyl}}_\epsilon)^{-1}(\check{\mathcal{K}}_{D,Q})\,, & \check{\mathcal{K}}^{<,\mathrm{cone}}_{\hat{D},\hat{Q}} &= (\Xi^{\mathrm{cone}}_\epsilon)^{-1}(\check{\mathcal{K}}_{D,Q})\,,\\
\check{\mathcal{K}}^{<,\mathrm{cyl}}(\hat{D},\hat{Q}) &= (\Xi^{\mathrm{cyl}}_\epsilon)^{-1}(\check{\mathcal{K}}(D,Q))\,, & \check{\mathcal{K}}^{<,\mathrm{cone}}(\hat{D},\hat{Q}) &= (\Xi^{\mathrm{cone}}_\epsilon)^{-1}(\check{\mathcal{K}}(D,Q))\;.
\end{aligned}
$$

Therefore, $(\check{\mathcal{K}}^{<,\mathrm{cyl}}_{\hat{D},\hat{Q}}, \blacklozenge^{\mathrm{cyl}}_\epsilon)$, $(\check{\mathcal{K}}^{<,\mathrm{cone}}_{\hat{D},\hat{Q}}, \blacklozenge^{\mathrm{cone}}_\epsilon)$ and $(\check{\mathcal{K}}^{<,\mathrm{cyl}}(\hat{D},\hat{Q}), \blacklozenge^{\mathrm{cyl}}_\epsilon)$, $(\check{\mathcal{K}}^{<,\mathrm{cone}}(\hat{D},\hat{Q}), \blacklozenge^{\mathrm{cone}}_\epsilon)$ provide isomorphic models for the $\mathbb{K}$-algebras $(\check{\mathcal{K}}_{D,Q}, \diamond)$ and $(\check{\mathcal{K}}(D,Q), \diamond)$, respectively. The collection of isomorphic models of the latter $\mathbb{K}$-algebras which were discussed in this section is summarized in the two commutative diagrams:

$$
\begin{tikzcd}
 & (\check{\mathcal{K}}^{<,\mathrm{cyl}}_{\hat{D},\hat{Q}}, \blacklozenge^{\mathrm{cyl}}_\epsilon) \arrow[rr, shift left, "P^{\mathrm{cyl}}_\epsilon"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \arrow[dd, dotted, shift left, "\Xi^{\mathrm{cyl}}_\epsilon"] & & (\check{\mathcal{K}}^{\mathrm{cyl}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cyl}}) \arrow[ll, shift left, "2P_<"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \arrow[dd, shift left, "2P_\perp"] \\
(\check{\mathcal{K}}^{<,\mathrm{cone}}_{\hat{D},\hat{Q}}, \blacklozenge^{\mathrm{cone}}_\epsilon) \arrow[ur, shift left, "r^{-\mathcal{E}}"] \arrow[rr, shift left, "P^{\mathrm{cone}}_\epsilon"] \arrow[dd, shift left, "\Xi^{\mathrm{cone}}_\epsilon"] & & (\check{\mathcal{K}}^{\mathrm{cone}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cone}}) \arrow[ll, shift left, "2P_<"] \arrow[ur, shift left, "r^{-\mathcal{E}}"] \arrow[dd, shift left, "2P_\perp"] & \\
 & (\check{\mathcal{K}}_{D,Q}, \diamond) \arrow[uu, dotted, shift left, "(\Xi^{\mathrm{cyl}}_\epsilon)^{-1}"] \arrow[rr, dotted, shift left, "\Pi^*"] \arrow[dl, dotted, shift left, "\mathrm{id}"] & & (\check{\mathcal{K}}_{\hat{D},\hat{Q}}, \diamond^{\mathrm{cyl}}) \arrow[ll, dotted, shift left, "j^*"] \arrow[uu, shift left, "P^{\mathrm{cyl}}_\epsilon"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \\
(\check{\mathcal{K}}_{D,Q}, \diamond) \arrow[uu, shift left, "(\Xi^{\mathrm{cone}}_\epsilon)^{-1}"] \arrow[ur, dotted, shift left, "\mathrm{id}"] \arrow[rr, shift left, "r^{\mathcal{E}} \circ\, \Pi^*"] & & (r^{\mathcal{E}}(\check{\mathcal{K}}_{\hat{D},\hat{Q}}), \diamond^{\mathrm{cone}}) \arrow[ll, shift left, "j^* \circ\, r^{-\mathcal{E}}"] \arrow[uu, shift left, "P^{\mathrm{cone}}_\epsilon"] \arrow[ur, shift left, "r^{-\mathcal{E}}"] &
\end{tikzcd}
$$

and:

$$
\begin{tikzcd}
 & (\check{\mathcal{K}}^{<,\mathrm{cyl}}(\hat{D},\hat{Q}), \blacklozenge^{\mathrm{cyl}}_\epsilon) \arrow[rr, shift left, "P^{\mathrm{cyl}}_\epsilon"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \arrow[dd, dotted, shift left, "\Xi^{\mathrm{cyl}}_\epsilon"] & & (\check{\mathcal{K}}^{\mathrm{cyl}}(\hat{D},\hat{Q}), \diamond^{\mathrm{cyl}}) \arrow[ll, shift left, "2P_<"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \arrow[dd, shift left, "2P_\perp"] \\
(\check{\mathcal{K}}^{<,\mathrm{cone}}(\hat{D},\hat{Q}), \blacklozenge^{\mathrm{cone}}_\epsilon) \arrow[ur, shift left, "r^{-\mathcal{E}}"] \arrow[rr, shift left, "P^{\mathrm{cone}}_\epsilon"] \arrow[dd, shift left, "\Xi^{\mathrm{cone}}_\epsilon"] & & (\check{\mathcal{K}}^{\mathrm{cone}}(\hat{D},\hat{Q}), \diamond^{\mathrm{cone}}) \arrow[ll, shift left, "2P_<"] \arrow[ur, shift left, "r^{-\mathcal{E}}"] \arrow[dd, shift left, "2P_\perp"] & \\
 & (\check{\mathcal{K}}(D,Q), \diamond) \arrow[uu, dotted, shift left, "(\Xi^{\mathrm{cyl}}_\epsilon)^{-1}"] \arrow[rr, dotted, shift left, "\Pi^*"] \arrow[dl, dotted, shift left, "\mathrm{id}"] & & (\check{\mathcal{K}}(\hat{D},\hat{Q}), \diamond^{\mathrm{cyl}}) \arrow[ll, dotted, shift left, "j^*"] \arrow[uu, shift left, "P^{\mathrm{cyl}}_\epsilon"] \arrow[dl, shift left, "r^{\mathcal{E}}"] \\
(\check{\mathcal{K}}(D,Q), \diamond) \arrow[uu, shift left, "(\Xi^{\mathrm{cone}}_\epsilon)^{-1}"] \arrow[ur, dotted, shift left, "\mathrm{id}"] \arrow[rr, shift left, "r^{\mathcal{E}} \circ\, \Pi^*"] & & (r^{\mathcal{E}}(\check{\mathcal{K}}(\hat{D},\hat{Q})), \diamond^{\mathrm{cone}}) \arrow[ll, shift left, "j^* \circ\, r^{-\mathcal{E}}"] \arrow[uu, shift left, "P^{\mathrm{cone}}_\epsilon"] \arrow[ur, shift left, "r^{-\mathcal{E}}"] &
\end{tikzcd}
$$

3 Application to $\mathcal{N} = 2$ flux compactifications of M-theory on eight-manifolds

In this section, we apply our methods to the study of the most general $\mathcal{N} = 2$ warped compactification of eleven-dimensional supergravity on eight-manifolds down to an AdS_3 space. After some basic preparations in subsection 3.1, subsection 3.2 explains how the cone formalism of section 2 can be applied to this example and gives a brief explanation of the reasons for relying on the cone construction. Subsection 3.3 gives our translation of the generalized Killing pinor equations into a system of algebraic and differential constraints on differential forms defined on the cone as well as a brief analysis of the structure of those equations. While these equations serve only an illustrative purpose in the present paper, they will be analyzed in detail in upcoming work.

3.1 Preparations

As in [3, 4, 7], we start with eleven dimensional supergravity on an 11-manifold endowed with a spinnable Lorentzian metric of 'mostly plus' signature. As in loc. cit., we consider compactification down to an AdS_3 space of cosmological constant $\Lambda = -8\kappa^2$, where κ is a positive real parameter — this includes the Minkowski case as the limit $\kappa \to 0$. Thus $\tilde{M} = N \times M$, where N is an oriented 3-manifold diffeomorphic with $\mathbb{R}^3$ and carrying the AdS_3 metric while M is an oriented Riemannian eight-manifold whose metric we denote by g. The metric on $\tilde{M}$ is a warped product whose warp factor Δ is a smooth function defined on M. For the field strength $\tilde{G}$, we use the ansatz:

$$\tilde{G} = e^{3\Delta} G \quad \text{with} \quad G = \mathrm{vol}_3 \wedge f + F\,,$$

where $f = f_m e^m \in \Omega^1(M)$, $F = \frac{1}{4!} F_{mnpq} e^{mnpq} \in \Omega^4(M)$ and vol_3 is the volume form of N. Small Latin indices from the middle of the alphabet run from 1 to 8 and correspond to a choice of frame on M. For the eleven-dimensional supersymmetry generator $\tilde{\eta}$, we use the ansatz:

$$\tilde{\eta} = e^{\frac{\Delta}{2}} \eta \quad \text{with} \quad \eta = \psi \otimes \xi\,,$$

where ξ is a real pinor of spin $1/2$ on the internal space M and ψ is a real pinor on the AdS_3 space N. Mathematically, ξ is a section of the pinor bundle of M, which is a real vector bundle of rank 16 defined on M, carrying a fiberwise representation of the Clifford algebra $\mathrm{Cl}(8, 0)$. Since $p - q \equiv_8 0$ for $p = 8$ and $q = 0$, this corresponds to the simple normal case of [7]. In particular, the corresponding morphism $\gamma : (\wedge T^*M, \diamond) \to (\mathrm{End}(S), \circ)$ of bundles of algebras is an isomorphism, i.e. it is bijective on the fibers. We set $\gamma^m = \gamma(e^m)$ and $\gamma^{(9)} := \gamma^1 \circ \ldots \circ \gamma^8$ for some local orthonormal frame e^m of M. In dimension eight with Euclidean signature, there exists an admissible [21, 22] (and thus $\mathrm{Spin}(8)$-invariant) bilinear pairing $\mathscr{B}$ on the pin bundle S, which is a scalar product. Assuming that ψ is a Killing pinor on the AdS_3 space, the supersymmetry condition amounts to the following *constrained generalized Killing (CGK)* pinor equations [7] for ξ:

$$D_m \xi = 0\,, \quad Q\xi = 0\,, \tag{3.1}$$

where D_m is a linear connection on S and $Q \in \Gamma(M, \mathrm{End}(S))$ is a globally-defined endomorphism of the vector bundle S. As in [3, 4] (and in contradistinction with [16]) *we do not require that ξ has definite chirality*. As we shall see in a moment, this seemingly trivial generalization has drastic consequences, leading to a problem which is technically much harder than that solved in the celebrated work of [16]. The space of solutions of (3.1) is a finite-dimensional $\mathbb{R}$-linear subspace $\mathcal{K}(D,Q)$ of the space $\Gamma(M,S)$ of smooth sections of S. The problem of interest is to find those metrics and fluxes on M for which some fixed amount of supersymmetry is preserved in three dimensions, i.e. for which the space $\mathcal{K}(D,Q)$ has some given non-vanishing dimension, which we denote by s. The case $s=1$ (which corresponds to $\mathcal{N}=1$ supersymmetry in three dimensions) was studied in [3, 4] and reconsidered in [7] by using geometric algebra techniques. The case $s=2$ (which leads to $\mathcal{N}=2$ supersymmetry in three dimensions) was studied in [16], but considering only Majorana-*Weyl* solutions ξ of (3.1), i.e. only the case when $\mathcal{K}_{D,Q}$ is a subspace of the kernel $\mathcal{K}(\mathrm{id}_S - \gamma_{(9)})$ or of the kernel $\mathcal{K}(\mathrm{id}_S + \gamma_{(9)})$. Here, we consider the case when no extraneous chirality constraint is imposed on the solutions of (3.1).

Direct computation gives the following expressions for D and Q (see [3, 7]):

$$D_m = \nabla_m^S + A_m\,, \quad A_m = \frac{1}{4} f_p \gamma_m{}^p \gamma^{(9)} + \frac{1}{24} F_{mpqr}\gamma^{pqr} + \kappa\gamma_m\gamma^{(9)} \tag{3.2}$$

and

$$Q = \frac{1}{2}\gamma^m \partial_m \Delta - \frac{1}{288} F_{mpqr}\gamma^{mpqr} - \frac{1}{6} f_p \gamma^p \gamma^{(9)} - \kappa\gamma^{(9)}\,. \tag{3.3}$$

In the present paper, we are interested in the case $s=2$ ($\mathcal{N}=2$ supersymmetry in three dimensions), so we require that (3.1) admits *two* linearly independent solutions ξ_1 and ξ_2. The formed-valued pinor bilinears $\check{\boldsymbol{E}}^{(k)}_{\xi_i,\xi_j} = \frac{1}{k!}\check{\boldsymbol{E}}^{(k)}_{m_1\ldots m_k}(\xi_i,\xi_j) e^{m_1\ldots m_k} \in \Omega^k(M)$ with $i,j=1,2$ are constrained by Fierz identities which play a crucial role below. As we shall see in a moment, these identities are much more involved in our case (even after reformulating them on the cone) than the identities which were encountered in [3] and [4]. The translation of (3.1) into equations on the differential forms $\check{\boldsymbol{E}}^{(k)}_{\xi_i,\xi_j}$ could be achieved starting from the following equivalent reformulation of the algebraic constraints $Q\xi_1 = Q\xi_2 = 0$:

$$\mathscr{B}(\xi_i, \left(Q^t \gamma_{m_1\ldots m_k} \pm \gamma_{m_1\ldots m_k} Q\right)\xi_j) = 0$$

and treating the differential constraints $D_m\xi_1 = D_m\xi_2 = 0$ through the method outlined in [3]. This direct approach due to [3] is discussed in detail in the appendices of [7], where it was also shown how that method is equivalent with the formalism developed in loc. cit. As it turns out, the direct approach is computationally quite impractical in our case and we have to rely on the methods and techniques of [7].

3.2 The cone construction

Before attempting to solve (3.1), one can ask whether the mere assumption of existence of two independent solutions ξ_1, ξ_2 provides some useful constraints on the geometry. The D-flatness conditions $D_m\xi_1 = D_m\xi_2 = 0$ imply that the values of the sections ξ_1,ξ_2 at two different points x,y on the internal manifold are related through the parallel transport of the connection A_m, which is an invertible linear operator between the fibers of S at x and y. In turn, this shows that two solutions which are linearly independent over $\mathbb{R}$ as sections of S must be linearly independent everywhere, i.e. the vectors $\xi_1(x), \xi_2(x) \in S_x$ must — for any point x of M — be linearly independent in the fiber $S_x \approx \mathbb{R}^{16}$ (and hence determine a point $(\xi_1(x), \xi_2(x))$ in the second Stiefel manifold $V_2(S_x)$ of S_x). Using this observation, one finds that solutions of (3.1) can be classified according to the orbit passing through $(\xi_1(x), \xi_2(x))$ of the action (induced by restricting γ_x) of the group $\mathrm{Spin}(8) \subset \mathrm{Cl}(8,0) \approx (\wedge T_x^* M, \diamond_x)$ on $V_2(S_x)$ — an orbit which is independent of x up to the action induced by the parallel transport of D. However, it turns out that the action of Spin(8) on the second Stiefel manifold of $\mathbb{R}^{16}$ fails to be transitive, which leads to complications when attempting to classify solutions in this manner. In particular (and unlike what happens in many other cases), two generic solutions of (3.1) fail to determine a *global* reduction of the structure group $\mathrm{SO}(8,\mathbb{R})$ of (M,g) — a phenomenon which (as discussed in [4]) also occurs for the case $s=1$ (the case of $\mathcal{N}=1$ supersymmetry in three dimensions). Due to this fact, it is convenient instead to consider ξ_1 and ξ_2 up to an action (induced by restricting γ_x) of the group Spin(9) — which can be viewed in a natural manner as a subgroup of $\mathrm{Cl}(8,0)$. As pointed out in [4], this Spin(9) action can be geometrized by introducing an extra dimension — for example, by passing to the metric cylinder (as in [4]) or to the metric cone (as we shall do below) over M. The fact that certain aspects of the simplest flux compactifications (such as Freund-Rubin compactifications on squashed spheres) can be better understood by passage to the metric cone is of course well-known — as is the fact that the (ordinary) Killing pinor equations of a Riemannian manifold can be reformulated as parallel pinor equations by passage to the metric cone (see [15]). In *general* flux compactifications, the simplification which one obtains through this construction is less drastic, though still quite useful both from a computational and conceptual standpoint.

Let us therefore consider the metric cone $(\hat{M}, g_{\mathrm{cone}})$ (cf. section 2) and the lift of D to the connections $\hat{D}$ on the pin bundle $\hat{S}$ of $\hat{M}$ (see subsection 2.7). Since the metric cone over (M,g) has signature $(9,0)$ and since $9-0 \equiv_8 1$, the Clifford algebra $\mathrm{Cl}(9,0)$ corresponds to the normal non-simple case discussed in [7]. We have two inequivalent[2] choices for the fiberwise pin representation of the Kähler-Atiyah bundle of $\hat{M}$, which are distinguished by the signature $\epsilon \in \{-1,1\}$. Both choices correspond to representations which are surjective but non-injective on each of the fibers $(\wedge T_x^* \hat{M}, \diamond_x) \approx \mathrm{Cl}(9,0)$. As in section 2, the pin bundle $\hat{S}$ of $\hat{M}$ can be constructed as the pullback of S through the natural projection of $\hat{M}$ to M, while the morphism $\gamma_{\mathrm{cone}} : (\wedge T^* \hat{M}, \diamond^{\mathrm{cone}}) \to (\mathrm{End}(\hat{S}), \circ)$ is completely determined by the morphism $\gamma : (\wedge T^* M, \diamond) \to (\mathrm{End}(S), \circ)$ once the signature ϵ has been chosen. In the following, we shall work with the choice $\epsilon = +1$. Setting $d=8$, $\epsilon = +1$ and $\rho = 2\kappa$ in the equations of subsection 2.3 and rescaling the metric on M as $g \to (2\kappa)^2 g$ (*without* changing the local orthonormal frame e_a^{cone} of the cone or the local orthonormal frame e_m of (M,g)) gives $\hat{D}_a = \nabla_a^{\hat{S},\mathrm{cone}} + A_a^{\mathrm{cone}}$, where:

$$\begin{aligned} \nabla_{\partial_r}^{\hat{S},\mathrm{cone}} &= \mathcal{L}_r^S, & \nabla_{e_m}^{\hat{S},\mathrm{cone}} &= \nabla_{e_m}^S + \kappa \gamma_{m9}\,, \\ A_9^{\mathrm{cone}} &= 0, & A_m^{\mathrm{cone}} &= \frac{1}{4} f^p \gamma_{mp9} + \frac{1}{24} F_{mpqr} \gamma^{pqr}\,, \end{aligned} \tag{3.4}$$

[2] Inequivalent in the sense of the representation theory of Clifford algebras.

where $\mathcal{L}_r^S$ in the right hand side denotes the Kosmann-Schwarzbach derivative [23, 24] on sections of $\hat{S}$, taken with respect to the vector field ∂_r. Here and below, indices from the middle of the Latin alphabet run from 1 to 8 and those from the beginning of the Latin alphabet run from 1 to 9. The latter correspond to frames e_a^{cyl} and e_a^{cone} chosen as in (2.68).

Notational convention. We do not indicate some pullbacks explicitly — in particular, we use the same notation for $\mathscr{B}$, γ^m and for their pullbacks $\hat{\mathscr{B}}$, $\hat{\gamma}^m = \gamma_*^m$ from S to $\hat{S}$, with a similar convention for differential forms. The bilinear pairing has the properties $\sigma_{\mathscr{B}} = +1$ and $\epsilon_{\mathscr{B}} = +1$.

As in subsection 2.3, the generalized Killing pinor equations $D_m\xi = 0$ ($m = 1 \ldots 8$) for pinors ξ defined on M amount to the flatness conditions $\hat{D}_a\hat{\xi} = 0$ ($a = 1 \ldots 9$) for pinors $\hat{\xi}$ defined on $\hat{M}$. Indeed, the last of the flatness equations $\hat{D}_9\hat{\xi} = 0$ is equivalent with the requirement that the section $\hat{\xi}$ of $\hat{S}$ is the pullback of some section ξ of S through the natural projection map from $\hat{M}$ to M, while the remaining equations amount to the generalized Killing conditions $D_m\xi = 0$ for the pinor ξ defined on the manifold M. Also recall from section 2 that the algebraic constraints are equivalent with the following equations for $\hat{\xi}$:

$$\hat{Q}\hat{\xi} = 0\,,$$

where $\hat{Q} \in \Gamma(\hat{M}, \mathrm{End}(\hat{S}))$ is the pullback of $Q \in \Gamma(M, \mathrm{End}(S))$ to $\hat{M}$. With our notational conventions (in which we won't explicitly indicate the pullback), $\hat{Q}$ has the same form (3.3) as Q in the appropriate local frame on the cone.

3.3 The $\mathbb{K}$-algebra of constrained generalized Killing forms

For the reasons outlined above, we consider the CGK pinor equations formulated on the metric cone over M. As explained in [7] (see also subsection 2.11), we realize the algebra $(\Omega^{+,\mathrm{cone}}(\hat{M}), \diamond^{\mathrm{cone}})$ (the effective domain of definition of γ_{cone}) as the algebra $(\Omega^{<}(\hat{M}), \blacklozenge_+^{\mathrm{cone}})$. We have $\Omega^{<}(\hat{M}) = \oplus_{k=0}^{4}\Omega^k(\hat{M})$, so we are interested in pinor bilinears $\check{\boldsymbol{E}}^{(k)}_{\hat{\xi}_1,\hat{\xi}_2}$ with $k = 0 \ldots 4$ for two independent solutions $\hat{\xi}_1, \hat{\xi}_2$ of the CGK pinor equations lifted to the cone:

$$\hat{D}_a\hat{\xi} = \hat{Q}\hat{\xi} = 0\,, \tag{3.5}$$

which are equivalent with the original CGK pinor equations on M.

To extract the translation of these equations into constraints on differential forms, we implemented certain procedures within the package `Ricci` [17] for tensor computations in `Mathematica`®. We also implemented similar procedures in `Cadabra` [18]. The dequantizations:

$$\check{A}_a^{\mathrm{cone}} = \gamma_{\mathrm{cone}}^{-1}(A_a^{\mathrm{cone}}) \in \Omega^{<}(\hat{M})\,, \quad \check{Q}^{\mathrm{cone}} = \gamma_{\mathrm{cone}}^{-1}(\hat{Q}) \in \Omega^{<}(\hat{M})\,,$$

of A^{cone} and $\hat{Q}$ are given by $\check{A}_9^{\mathrm{cone}} = 0$ and (recall that $\theta \stackrel{\mathrm{def.}}{=} \mathrm{d}r = e^9_{\mathrm{cone}}$):

$$\begin{aligned}
\check{A}_m^{\mathrm{cone}} &= \frac{1}{4}\iota^{\mathrm{cone}}_{e_m^{\mathrm{cone}}}F_{\mathrm{cone}} + \frac{1}{4}(e_m^{\mathrm{cone}})_{\#,\mathrm{cone}} \wedge f_{\mathrm{cone}} \wedge \theta \quad \in \Omega^{<}(\hat{M})\,,\\
\check{Q}^{\mathrm{cone}} &= \frac{1}{2}r\mathrm{d}\Delta - \frac{1}{6}f_{\mathrm{cone}} \wedge \theta - \frac{1}{12}F_{\mathrm{cone}} - \kappa\theta \quad \in \Omega^{<}(\hat{M})\,,
\end{aligned}$$

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while the $\mathscr{B}$-transpose of $\hat{Q}$ dequantizes to the reversion of $\check{Q}^{\rm cone}$:

$$\hat{\tau}(\check{Q}^{\rm cone}) = \frac{1}{2} r \mathrm{d}\Delta + \frac{1}{6} f_{\rm cone} \wedge \theta - \frac{1}{12} F_{\rm cone} - \kappa\theta \ .$$

Here, $\hat{\tau}$ is the main anti-automorphism [7] of $(\Omega(M), \diamond^{\rm cone})$ (which coincides with the main anti-automorphism of $(\Omega(M), \diamond^{\rm cyl})$). The forms $f_{\rm cone}$ and $F_{\rm cone}$ above are the cone lifts (see definition (2.30)) of f and F respectively, while (in accordance with our notational conventions) Δ stands for the pullback $\Pi^*(\Delta) = \Delta \circ \Pi$, even though the notation does not show this explicitly.

A basis for the space spanned by $\check{\boldsymbol{E}}^{(k),\rm cone}_{\hat{\xi}_1,\hat{\xi}_2} \stackrel{\rm def.}{=} \frac{1}{k!}(\epsilon_{\mathscr{B}})^k \mathscr{B}(\hat{\xi}_1, \hat{\gamma}_{a_1 \ldots a_k} \hat{\xi}_2) e^{a_1}_{\rm cone} \wedge \ldots \wedge e^{a_k}_{\rm cone} \in \Omega^{<}(\hat{M})$ (of rank at most 4) which can be constructed on the cone from $\hat{\xi}_1$ and $\hat{\xi}_2$ is given by (where we have raised all indices using the cone metric in order to avoid notational clutter):

$$\begin{aligned}
V_1^a &= \mathscr{B}(\hat{\xi}_1, \hat{\gamma}^a \hat{\xi}_1)\,, & V_2^a &= \mathscr{B}(\hat{\xi}_2, \hat{\gamma}^a \hat{\xi}_2)\,, & V_3^a &= \mathscr{B}(\hat{\xi}_1, \hat{\gamma}^a \hat{\xi}_2)\,, \\
K^{ab} &= \mathscr{B}(\hat{\xi}_1, \hat{\gamma}^{ab} \hat{\xi}_2), & \Psi^{abc} &= \mathscr{B}(\hat{\xi}_1, \hat{\gamma}^{abc} \hat{\xi}_2)\,, & & \\
\Phi_1^{abce} &= \mathscr{B}(\hat{\xi}_1, \hat{\gamma}^{abce} \hat{\xi}_1)\,, & \Phi_2^{abce} &= \mathscr{B}(\hat{\xi}_2, \hat{\gamma}^{abce} \hat{\xi}_2)\,, & \Phi_3^{abce} &= \mathscr{B}(\hat{\xi}_1, \hat{\gamma}^{abce} \hat{\xi}_2)\,,
\end{aligned} \tag{3.6}$$

Here and below, we have taken $\hat{\xi}_1$ and $\hat{\xi}_2$ to form an orthonormal basis of the $\mathbb{R}$-vector space $\mathcal{K}(\hat{D}, \hat{Q})$ on the cone:

$$\mathscr{B}(\hat{\xi}_i, \hat{\xi}_j) = \delta_{ij}\,, \quad \forall i,j = 1,2 \ .$$

To arrive at (3.6), we used the identity $\mathscr{B}(\hat{\xi}_i, \hat{\gamma}^{a_1 \ldots a_k} \hat{\xi}_j) = (-1)^{\frac{k(k-1)}{2}} \mathscr{B}(\hat{\xi}_j, \hat{\gamma}^{a_1 \ldots a_k} \hat{\xi}_i)$, which follows from $(\hat{\gamma}_a)^t = \hat{\gamma}_a$ and implies that certain of the forms $\check{\boldsymbol{E}}^{(k),\rm cone}_{\hat{\xi}_i,\hat{\xi}_j}$ vanish identically.

Notational convention. From now on — in order to avoid notational clutter — we shall suppress the superscripts and subscripts "cone". In particular, we shall denote the cone lifts $F_{\rm cone} = r^4 \Pi^*(F)$ and $f_{\rm cone} = r\Pi^*(f)$ simply by F and f. We remind the reader of our notations in [7] for the basis elements $\check{E}_{ij}$ of the algebra $(\Omega^+(\hat{M}), \diamond)$ on the cone:

$$\check{E}_{ij} \stackrel{\rm def.}{=} \check{E}_{\xi_i,\xi_j} = \frac{1}{2^{[\frac{d+1}{2}]}} \check{\boldsymbol{E}}_{\xi_i,\xi_j} = \frac{1}{2^{[\frac{d+1}{2}]}} \sum_{k=0}^{d} \check{\boldsymbol{E}}^{(k)}_{\xi_i,\xi_j} \ \in \Omega^+(M)\,,$$

where we know that $\check{E}_{ij}$ are twisted selfdual forms, thus $\check{E}_{ij} = \check{E}^<_{ij} + \tilde{*}\check{E}^<_{ij}$, where $\check{E}^<_{ij} \in \Omega^<(\hat{M})$. Hence the basis elements $\check{E}^<_{ij}$ of the truncated Fierz algebra $(\Omega^<(\hat{M}), \blacklozenge_+)$ read:

$$\check{E}^<_{ij} = \frac{1}{2^{[\frac{d+1}{2}]}} \sum_{k=0}^{[\frac{d}{2}]} \check{\boldsymbol{E}}^{(k)}_{\xi_i,\xi_j} \ \in \Omega^<(M)\,,$$

where $d = 9$. With the notations and conventions above, the truncated model of the flat Fierz $\mathbb{K}$-algebra $(\check{\mathcal{K}}^<(\hat{D}, \hat{Q}), \blacklozenge_+)$ on the cone admits the basis:

$$\begin{aligned}
\check{E}^<_{12} &= \frac{1}{32}(V_3 + K + \Psi + \Phi_3)\,, & \check{E}^<_{21} &= \frac{1}{32}(V_3 - K - \Psi + \Phi_3)\,, \\
\check{E}^<_{11} &= \frac{1}{32}(1 + V_1 + \Phi_1)\,, & \check{E}^<_{22} &= \frac{1}{32}(1 + V_2 + \Phi_2)
\end{aligned} \tag{3.7}$$

and can be generated by two elements (see subsection 5.10 of [7]), which we choose to be:

$$\check{E}^{<}_{12} = \frac{1}{32}(V_3 + K + \Psi + \Phi_3) \quad \text{and} \quad \check{E}^{<}_{21} = \hat{\tau}(\check{E}_{12}) = \frac{1}{32}(V_3 - K - \Psi + \Phi_3)\ .$$

In order to avoid notational clutter, we shall henceforth use $\blacklozenge$ instead of $\blacklozenge_+$. As exlained in general in [7], the Fierz relations for the truncated model amount to:

$$\check{E}^{<}_{ij}\blacklozenge\check{E}^{<}_{kl} = \frac{1}{2}\delta_{jk}\check{E}_{il}\,, \quad \forall i,j,k,l = 1,2\,, \tag{3.8}$$

while the ideal of relations corresponding to $\check{E}^{<}_{12}$ and $\check{E}^{<}_{21}$ is generated by:

$$\check{E}^{<}_{12}\blacklozenge\check{E}_{12} = 0 \qquad \left(\Longleftrightarrow \hat{\tau}(\check{E}^{<}_{12})\blacklozenge\hat{\tau}(\check{E}_{12}) = 0\right), \tag{3.9}$$

$$\check{E}^{<}_{12}\blacklozenge\hat{\tau}(\check{E}^{<}_{12})\blacklozenge\check{E}^{<}_{12} = \frac{1}{4}\check{E}^{<}_{12} \quad \left(\Longleftrightarrow \hat{\tau}(\check{E}^{<}_{12})\blacklozenge\check{E}_{12}\blacklozenge\hat{\tau}(\check{E}^{<}_{12}) = \frac{1}{4}\hat{\tau}(\check{E}^{<}_{12})\right). \tag{3.10}$$

On the other hand, the algebraic constraints in (3.5) amount to the following two relations for $\check{E}^{<}_{12}$:

$$\check{Q}\blacklozenge\check{E}_{12} \mp \check{E}_{12}\blacklozenge\hat{\tau}(\check{Q}) = 0\,, \tag{3.11}$$

while the differential constraints of (3.5) give the equations $\check{D}^{\mathrm{ad}}_a\check{E}^{<}_{12} = 0 \Leftrightarrow \check{D}^{\mathrm{ad}}_a\check{E}^{<}_{21} = 0$, which in turn imply:

$$\mathrm{d}\check{E}^{<}_{12} = e^a \wedge \nabla_a\check{E}^{<}_{12} = -2e^a \wedge [\check{A}_a, \check{E}^{<}_{12}]_{-,\blacklozenge}\ . \tag{3.12}$$

As explained in subsection 5.10 of [7], it is enough to consider the constraints (3.11) and (3.12) for the generators $\check{E}^{<}_{12}$ and $\check{E}^{<}_{21} = \hat{\tau}(\check{E}^{<}_{12})$, since the corresponding constraints for $\check{E}^{<}_{11} = 2\check{E}^{<}_{12}\blacklozenge\check{E}^{<}_{21}$ and $\check{E}^{<}_{22} = 2\check{E}^{<}_{21}\blacklozenge\check{E}^{<}_{12}$ follow from those.

Algebraic constraints. Using the procedures which we have implemented and the package `Ricci` for tensor computations in `Mathematica`® (see [17]), we find that the first equation (with the minus sign) in (3.11) amounts to the following system when separated on ranks:

$$\begin{aligned}
\iota_{f\wedge\theta}K &= 0\,,\\
r\iota_{\mathrm{d}\Delta}K + \frac{1}{3}\iota_{f\wedge\theta}\Psi - \frac{1}{6}\iota_{\Psi}F - 2\kappa\ \iota_{\theta}K &= 0\,,\\
\frac{1}{3}\iota_{f\wedge\theta}\Phi_3 - \frac{1}{6}F \bigtriangleup_3 \Phi_3 + r(\mathrm{d}\Delta)\wedge V_3 + 2\kappa\ V_3\wedge\theta &= 0\,,\\
r\iota_{\mathrm{d}\Delta}\Phi_3 - \frac{1}{3}V_3\wedge f\wedge\theta + \frac{1}{6}\iota_{V_3}F - \frac{1}{6}*(F \bigtriangleup_1 \Phi_3) + \frac{1}{3}*(f\wedge\theta\wedge\Phi_3) - 2\kappa\ \iota_{\theta}\Phi_3 &= 0\,,\\
r(\mathrm{d}\Delta)\wedge\Psi - \frac{1}{3}f\wedge\theta\wedge K - \frac{1}{6}K \bigtriangleup_1 F - \frac{1}{3}*(f\wedge\theta\wedge\Psi) + \frac{1}{6}*(\Psi \bigtriangleup_1 F) + 2\kappa\ \Psi\wedge\theta &= 0\,,
\end{aligned}$$

while the second equation (with the plus sign) in (3.11) amounts to:

$$
\begin{aligned}
-\frac{1}{6}\iota_F\Phi_3 + r\iota_{\mathrm{d}\Delta}V_3 - 2\kappa\ \iota_\theta V_3 &= 0\,,\\
\frac{1}{3}\iota_{V_3}(f\wedge\theta) - \frac{1}{6}*(F\wedge\Phi_3) &= 0\,,\\
r\iota_{\mathrm{d}\Delta}\Psi + \frac{1}{3}(f\wedge\theta)\bigtriangleup_1 K + \frac{1}{6}\iota_K F + \frac{1}{6}*(F\wedge\Psi) - 2\kappa\ \iota_\theta\Psi &= 0\,,\\
\frac{1}{3}(f\wedge\theta)\bigtriangleup_1 \Psi + \frac{1}{6}\Psi\bigtriangleup_2 F + \frac{1}{6}*(K\wedge F) + r(\mathrm{d}\Delta)\wedge K - 2\kappa\ K\wedge\theta &= 0\,,\\
\frac{1}{3}(f\wedge\theta)\bigtriangleup_1 \Phi_3 + \frac{1}{6}F\bigtriangleup_2 \Phi_3 - \frac{1}{6}*(F\wedge V_3) + *(r(\mathrm{d}\Delta)\wedge\Phi_3) - 2\kappa*(\Phi_3\wedge\theta) &= 0\,.
\end{aligned}
$$

Differential constraints. Using the same `Mathematica`® package, we can extract the differential constraints given by (3.12), which — when separated on ranks — amount to:

$$
\begin{aligned}
\mathrm{d}V_3 &= \Phi_3\bigtriangleup_3 F + 2\iota_{f\wedge\theta}\Phi_3\,,\\
\mathrm{d}K &= 2(f\wedge\theta)\bigtriangleup_1\Psi + 2\Psi\bigtriangleup_2 F\,,\\
\mathrm{d}\Psi &= 3F\bigtriangleup_1 K - F\bigtriangleup_3 *\Psi + 4*(f\wedge\theta\wedge\Psi) - 2f\wedge\theta\wedge K\,,\\
\mathrm{d}\Phi_3 &= -4F\wedge V_3 + e^m\wedge*((\iota_{e^m}F)\bigtriangleup_1\Phi_3) - e^m\wedge*(((e_m)_\#\wedge f\wedge\theta)\bigtriangleup_1\Phi_3)\,.
\end{aligned}
$$

According to our notational conventions, e^m in the equations above stands for e^m_{cone} while e_m stands for e_m^{cone}. Furthermore, $* \overset{\mathrm{def.}}{=} *_{\mathrm{cone}}$ is the (ordinary[3]) Hodge operator of (M, g_{cone}) and ι stands for ι^{cone}. The generalized products $\bigtriangleup_p \overset{\mathrm{def.}}{=} \bigtriangleup_p^{\mathrm{cone}}$ are constructed with the cone metric.

Fierz relations. Let us consider the Fierz identities (3.8) for the basis elements $\check{E}_{ij}$ $(i,j=1,2)$ of the truncated model of the flat Fierz algebra $(\check{\mathcal{K}}^<(\hat{D},\hat{Q},\blacklozenge)$:

$$
\begin{aligned}
&(\mathrm{F1}):\quad \check{E}^<_{12}\blacklozenge\check{E}^<_{12} = 0\,, &&(\mathrm{F2}):\quad \check{E}^<_{12}\blacklozenge\check{E}^<_{21} = \frac{1}{2}\check{E}^<_{11}\,, &&(3.13)\\
&(\mathrm{F3}):\quad \check{E}^<_{12}\blacklozenge\check{E}^<_{22} = \frac{1}{2}\check{E}^<_{12}\,, &&(\mathrm{F4}):\quad \check{E}^<_{12}\blacklozenge\check{E}^<_{11} = 0\,, &&(3.14)\\
&(\mathrm{F5}):\quad \check{E}^<_{11}\blacklozenge\check{E}^<_{11} = \frac{1}{2}\check{E}^<_{11}\,, &&(\mathrm{F6}):\quad \check{E}^<_{11}\blacklozenge\check{E}^<_{12} = \frac{1}{2}\check{E}^<_{12}\,, &&(3.15)\\
&(\mathrm{F7}):\quad \check{E}^<_{11}\blacklozenge\check{E}^<_{21} = 0\,, &&(\mathrm{F8}):\quad \check{E}^<_{11}\blacklozenge\check{E}^<_{22} = 0\,, &&(3.16)\\
&(\mathrm{F9}):\quad \check{E}^<_{21}\blacklozenge\check{E}^<_{12} = \frac{1}{2}\check{E}^<_{22}\,, &&(\mathrm{F10}):\quad \check{E}^<_{21}\blacklozenge\check{E}^<_{11} = \frac{1}{2}\check{E}^<_{21}\,, &&(3.17)\\
&(\mathrm{F11}):\quad \check{E}^<_{21}\blacklozenge\check{E}^<_{21} = 0\,, &&(\mathrm{F12}):\quad \check{E}^<_{21}\blacklozenge\check{E}^<_{22} = 0\,, &&(3.18)\\
&(\mathrm{F13}):\quad \check{E}^<_{12}\blacklozenge\check{E}^<_{11} = 0\,, &&(\mathrm{F14}):\quad \check{E}^<_{22}\blacklozenge\check{E}^<_{12} = 0\,, &&(3.19)\\
&(\mathrm{F15}):\quad \check{E}^<_{22}\blacklozenge\check{E}^<_{21} = \frac{1}{2}\check{E}^<_{21}\,, &&(\mathrm{F16}):\quad \check{E}^<_{22}\blacklozenge\check{E}^<_{22} = \frac{1}{2}\check{E}^<_{22}\,. &&(3.20)
\end{aligned}
$$

We note that some of these conditions are equivalent through reversion (namely (F1)⇔(F11), (F3)⇔(F15), (F4)⇔(F7), (F6)⇔(F10), (F8)⇔(F13) and (F12)⇔(F14), while

[3]As opposed to the *twisted* Hodge operator of [7].

relations (F2), (F5), (F9), (F16) are selfdual under reversion). After expanding the geometric product and separating ranks, we find independent relations only from certain rank components of (F1) — (F6), (F8), (F9), (F12) and (F16). Namely, equation (F1) (which coincides with (3.9)) takes the form:

$$(V_3 + K + \Psi + \Phi_3)\blacklozenge(V_3 + K + \Psi + \Phi_3) = 0$$

and gives the following relations when separated into rank components:

$$\begin{aligned}
-||K||^2 + ||\Phi_3||^2 - ||\Psi||^2 + ||V_3||^2 &= 0\,,\\
-2\iota_K\Psi + *(\Phi_3 \wedge \Phi_3) &= 0\,,\\
\iota_{V_3}\Psi - *(\Phi_3 \wedge \Psi) - \iota_K\Phi_3 &= 0\,,\\
K \wedge V_3 - *(K \wedge \Phi_3) - \Psi \bigtriangleup_2 \Phi_3 &= 0\,,\\
\Psi \bigtriangleup_1 \Psi - \Phi_3 \bigtriangleup_2 \Phi_3 + 2*(K \wedge \Psi) + 2*(V_3 \wedge \Phi_3) + K \wedge K &= 0\,.
\end{aligned}$$

Separating (F2) into rank components gives the following nontrivial relations:

$$\begin{aligned}
||K||^2 + ||\Phi_3||^2 + ||\Psi||^2 + ||V_3||^2 &= 16\,,\\
2\iota_K\Psi - 2\iota_{V_3}K - 2\iota_\Psi\Phi_3 + *(\Phi_3 \wedge \Phi_3) - 16V_1 &= 0\,,\\
-\Psi \bigtriangleup_1 \Psi - \Phi_3 \bigtriangleup_2 \Phi_3 - 2*(K \wedge \Psi) + 2*(V_3 \wedge \Phi_3) - K \wedge K - 16\Phi_1 - &\\
-2K \bigtriangleup_1 \Phi_3 - 2V_3 \wedge \Psi + 2*(\Psi \bigtriangleup_1 \Phi_3) &= 0
\end{aligned}$$

and similarly for (F3):

$$\begin{aligned}
\langle\Phi_2, \Phi_3\rangle + \langle V_2, V_3\rangle &= 0\,,\\
-\iota_{V_2}K - \iota_\Psi\Phi_2 + *(\Phi_2 \wedge \Phi_3) - 15V_3 &= 0\,,\\
-15K - V_2 \wedge V_3 - \Phi_2 \bigtriangleup_3 \Phi_3 - \iota_K\Phi_2 + \iota_{V_2}\Psi - *(\Phi_2 \wedge \Psi) &= 0\,,\\
K \wedge V_2 + \iota_{V_3}\Phi_2 - \iota_{V_2}\Phi_3 - *(K \wedge \Phi_2) - 15\Psi - *(\Phi_2 \bigtriangleup_1 \Phi_3) - \Psi \bigtriangleup_2 \Phi_2 &= 0\,,\\
-K \bigtriangleup_1 \Phi_2 - \Phi_2 \bigtriangleup_2 \Phi_3 + *(\Psi \bigtriangleup_1 \Phi_2) + *(V_2 \wedge \Phi_3) + *(V_3 \wedge \Phi_2) - 15\Phi_3 - V_2 \wedge \Psi &= 0\,,
\end{aligned}$$

for (F4):

$$\begin{aligned}
\langle\Phi_1, \Phi_3\rangle + \langle V_1, V_3\rangle &= 0\,,\\
-\iota_{V_1}K - \iota_\Psi\Phi_1 + *(\Phi_1 \wedge \Phi_3) + V_3 &= 0\,,\\
K - V_1 \wedge V_3 - \Phi_1 \bigtriangleup_3 \Phi_3 - \iota_K\Phi_1 + \iota_{V_1}\Psi - *(\Phi_1 \wedge \Psi) &= 0\,,\\
K \wedge V_1 + \iota_{V_3}\Phi_1 - \iota_{V_1}\Phi_3 - *(K \wedge \Phi_1) + \Psi - *(\Phi_1 \bigtriangleup_1 \Phi_3) - \Psi \bigtriangleup_2 \Phi_1 &= 0\,,\\
-K \bigtriangleup_1 \Phi_1 - \Phi_1 \bigtriangleup_2 \Phi_3 + *(\Psi \bigtriangleup_1 \Phi_1) + *(V_1 \wedge \Phi_3) + *(V_3 \wedge \Phi_1) + \Phi_3 - V_1 \wedge \Psi &= 0\,,
\end{aligned}$$

(F5):

$$\begin{aligned}
||\Phi_1||^2 + ||V_1||^2 &= 15\,,\\
*(\Phi_1 \wedge \Phi_1) - 14V_1 &= 0\,,\\
-\Phi_1 \bigtriangleup_2 \Phi_1 + 2*(V_1 \wedge \Phi_1) - 14\Phi_1 &= 0\,,
\end{aligned}$$

(F6) (not writing the rank 0 component, equal with the rank 0 component of (F4)):

$$\begin{aligned}
\iota_{V_1}K + \iota_\Psi\Phi_1 + *(\Phi_1\wedge\Phi_3) - 15V_3 &= 0\,,\\
-15K + V_1\wedge V_3 + \Phi_1 \bigtriangleup_3 \Phi_3 - \iota_K\Phi_1 + \iota_{V_1}\Psi - *(\Phi_1\wedge\Psi) &= 0\,,\\
K\wedge V_1 - \iota_{V_3}\Phi_1 + \iota_{V_1}\Phi_3 - *(K\wedge\Phi_1) - 15\Psi + *(\Phi_1 \bigtriangleup_1 \Phi_3) - \Psi \bigtriangleup_2 \Phi_1 &= 0\,,\\
K \bigtriangleup_1 \Phi_1 - \Phi_1 \bigtriangleup_2 \Phi_3 - *(\Psi \bigtriangleup_1 \Phi_1) + *(V_1\wedge\Phi_3) + *(V_3\wedge\Phi_1) - 15\Phi_3 + V_1\wedge\Psi &= 0\,,
\end{aligned}$$

(F8):

$$\begin{aligned}
1 + \langle\Phi_1,\Phi_2\rangle + \langle V_1, V_2\rangle &= 0\,,\\
*(\Phi_1\wedge\Phi_2) + V_1 + V_2 &= 0\,,\\
V_1\wedge V_2 + \Phi_1 \bigtriangleup_3 \Phi_2 &= 0\,,\\
\iota_{V_1}\Phi_2 - \iota_{V_2}\Phi_1 + *(\Phi_1 \bigtriangleup_1 \Phi_2) &= 0\,,\\
-\Phi_1 \bigtriangleup_2 \Phi_2 + *(V_1\wedge\Phi_2) + *(V_2\wedge\Phi_1) + \Phi_1 + \Phi_2 &= 0\,,
\end{aligned}$$

(F9) (not writing the rank 0 component, equal with the rank 0 component of (F2)):

$$\begin{aligned}
2\iota_K\Psi + 2\iota_{V_3}K + 2\iota_\Psi\Phi_3 + *(\Phi_3\wedge\Phi_3) - 16V_2 &= 0\,,\\
-\Psi \bigtriangleup_1 \Psi - \Phi_3 \bigtriangleup_2 \Phi_3 - 2*(K\wedge\Psi) - K\wedge K - 16\Phi_2 + 2*(V_3\wedge\Phi_3) + &\\
+2K \bigtriangleup_1 \Phi_3 + 2V_3\wedge\Psi - 2*(\Psi \bigtriangleup_1 \Phi_3) &= 0\,,
\end{aligned}$$

(F12):

$$\begin{aligned}
\langle\Phi_2,\Phi_3\rangle + \langle V_2, V_3\rangle &= 0\,,\\
\iota_{V_2}K + \iota_\Psi\Phi_2 + *(\Phi_2\wedge\Phi_3) + V_3 &= 0\,,\\
-\Phi_2 \bigtriangleup_3 \Phi_3 - \iota_{V_2}\Psi + *(\Phi_2\wedge\Psi) + \iota_K\Phi_2 - K - V_2\wedge V_3 &= 0\,,\\
-K\wedge V_2 + \iota_{V_3}\Phi_2 - \iota_{V_2}\Phi_3 + *(K\wedge\Phi_2) - \Psi - *(\Phi_2 \bigtriangleup_1 \Phi_3) + \Psi \bigtriangleup_2 \Phi_2 &= 0\,,\\
K \bigtriangleup_1 \Phi_2 - \Phi_2 \bigtriangleup_2 \Phi_3 - *(\Psi \bigtriangleup_1 \Phi_2) + *(V_2\wedge\Phi_3) + *(V_3\wedge\Phi_2) + \Phi_3 + V_2\wedge\Psi &= 0\,,
\end{aligned}$$

and finally for (F16):

$$\begin{aligned}
||\Phi_2||^2 + ||V_2||^2 &= 15\,,\\
*(\Phi_2\wedge\Phi_2) - 14V_2 &= 0\,,\\
-\Phi_2 \bigtriangleup_2 \Phi_2 + 2*(V_2\wedge\Phi_2) - 14\Phi_2 &= 0\ .
\end{aligned}$$

The system of equations given above can be studied by elimination. Its detailed analysis and implications are taken up in a forthcoming publication.

4 Conclusions and further directions

We studied the Kähler-Atiyah algebra of metric cones and cylinders and certain subalgebras thereof, constructing a number of isomorphic models which can be used to study the Kähler-Atiyah algebra of their unit sections and to lift generalized Killing equations, Fierz isomorphisms etc. from a (pseudo-)Riemannian manifold to its metric cylinder or cone. These results provide a toolkit for the study of generalized Killing spinor equations on (pseudo-)Riemannian manifolds, being especially relevant to problems which arise in flux compactifications. Our formulation is highly amenable to implementation in various symbolic computation systems specialized in tensor algebra, and we touched on two particular implementations which we have carried out using `Ricci` [17] and `Mathematica`® as well as `Cadabra` [18]. We illustrated our techniques for the case of the most general flux compactifications of M-theory which preserve $\mathcal{N} = 2$ supersymmetry in three dimensions, a class of compactifications whose most general members were not studied before — obtaining a complete description of the differential and algebraic constraints on pinor bilinears and uncovering the underlying algebraic structure. A detailed analysis of the resulting equations, geometry and physics is the subject of ongoing work.

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